Mechanics 1: Projectiles, Constrained Motion & Friction

We begin by introducing some new concepts and defining some new terms.

**Projectiles.** What’s a projectile? It’s an object that is “projected through space”. That is, an object (for us, this will be a particle of constant mass $m$), at some initial position and time, is given an initial velocity. We are interested in the motion thereafter, which is subject to the applied forces (e.g., gravity, air resistance, etc.). Classically, such problems have been phrased in terms of the flight of cannonballs, bullets, or thrown balls. However, an understanding of the motion of projectiles is fundamental to many current problems in science and engineering. For example, a topic of great recent interest has been concerned with trying to understand how certain rock fragments from the planet Mars made their way to Earth. It turns out that this is a result of the impact of an asteroid with the surface of Mars. The impact dislodged fragments of the planet’s surface, flinging them out into space. Their subsequent trajectories then came under the influence of the Earth’s gravitational field, pulling them onto the surface of the Earth.

**Weight and Acceleration Due to Gravity.** It has been experimentally determined that, near the earth’s surface, objects fall with a constant vertical acceleration, provided that air resistance is negligible. This acceleration is denoted by $g$, and is referred to as the *acceleration due to gravity* or the *gravitational acceleration*. It is approximately equal to $9.8 \text{m/sec}^2$.

Assuming that the earth’s surface is represented by the $x-y$ plane, the force due to gravity acting on a particle of mass $m$ is given by:

$$W = -mgk.$$  

This force, which is referred to as the *weight of the particle*, has magnitude $W = mg$, see Fig. 1.

![Figure 1](image)

We remark that this assumption of a flat earth is only approximately correct. Moreover, the force of gravity varies slightly over the surface of the earth. You will learn more about this when you learn about *Newton’s Law of Gravitation*.

**Constrained Motion.** In some cases a particle is forced to move along a curve or surface. This curve or surface is referred to as a *constraint*, and the resulting motion is called *constrained motion*.

The particle exerts a force on the constraint, and by Newton’s third law the constraint exerts a force on the particle. This force is called the *reaction force*, and is described by giving its components normal to the motion, denoted $N$, and parallel to the motion, denoted $f$. 
**Friction.** In the constrained motion of a particle a common force parallel to the motion of the particle is friction, which typically acts in the direction opposite to that of the motion (i.e., it acts in such a way as to slow the motion of the particle). Experimentally, it is found that the magnitude of the frictional force is proportional to the magnitude of the normal force, i.e.,

\[ f = \mu N, \]

where the proportionality constant, \( \mu \), is referred to as the coefficient of friction. Typically, its value depends on both the material of the particle and the constraint.

**Example.** Consider a projectile of constant mass \( m \) that is launched with initial speed \( v_0 \) at an angle \( \alpha \) with the horizontal. We will compute:

1. the position vector at any time,
2. the time to reach the highest point,
3. the maximum height reached,
4. the time of flight back to earth,
5. the range.

For an illustration of the geometry see Fig. 2.

![Figure 2:](image)

We will denote the position vector of the projectile at any time \( t \) by \( \mathbf{r} \), and the velocity at any time \( t \) by \( \mathbf{v} \). Of course, we begin by writing down Newton’s equations:

\[ m \frac{d^2 \mathbf{r}}{dt^2} = -m g \mathbf{k}, \]

which, after dividing both sides by the mass, become:

\[ \frac{d^2 \mathbf{r}}{dt^2} = -g \mathbf{k}, \quad \text{or} \quad \frac{dv}{dt} = -g \mathbf{k}. \]

The equation for \( \mathbf{v} \) is easily solved to give:

\[ \mathbf{v} = -gt \mathbf{k} + \mathbf{c}_1. \quad (1) \]

To evaluate the constant vector \( \mathbf{c}_1 \) we need to require our solution to satisfy the initial conditions. We assume that our initial velocity is in the \( y-z \) plane (why is it sufficient to assume this?) so that the initial conditions take the form:
\[ \mathbf{v}_0 = v_0 \cos \alpha \mathbf{j} + v_0 \sin \alpha \mathbf{k}. \] 

Evaluating (1) at \( t = 0 \) and equating the result to \( \mathbf{v}_0 \) gives \( c_1 = \mathbf{v}_0 \), and therefore we have:

\[ \mathbf{v} = v_0 \cos \alpha \mathbf{j} + (v_0 \sin \alpha - gt)\mathbf{k}. \] 

Now we can solve for the position vector. Writing \( \mathbf{v} = \frac{dr}{dt} \), substituting this into (3), and integrating from 0 to \( t \) (using the initial condition \( r(0) = 0 \)), gives:

\[ \mathbf{r} = (v_0 \cos \alpha)\mathbf{j} + \left( (v_0 \sin \alpha)t - \frac{g}{2}t^2 \right) \mathbf{k}. \]

or, in components,

\[ x = 0, \quad y = (v_0 \cos \alpha)t, \quad z = (v_0 \sin \alpha)t - \frac{g}{2}t^2. \]

So we see that if the projectile has initial velocity in the \( y-z \) plane, it remains in the \( y-z \) plane. Should we have been able to determine this without solving the equations of motion?

Next we compute the time to reach the highest point. What characterizes the highest point? The projectile goes up, then it comes down. In order to come down, the \( z \) component of velocity must change from positive to negative. Therefore, at the highest point, the \( z \) component of velocity vanishes, i.e.,

\[ v_0 \sin \alpha - gt = 0, \quad \text{and therefore} \quad t = \frac{v_0 \sin \alpha}{g}. \]

Next we compute the maximum height reached. What characterizes the maximum height? This is the same as the previous question. The height is a maximum when the \( z \) component of velocity vanishes. We computed the time for this to occur in the previous question. We then substitute this time into the expression for the \( z \) component of position given in (5) to obtain:

maximum height reached = \( (v_0 \sin \alpha) \left( \frac{v_0 \sin \alpha}{g} \right) - \frac{g}{2} \left( \frac{v_0 \sin \alpha}{g} \right)^2 = \frac{v_0^2 \sin^2 \alpha}{2g}. \)

Next we compute the time of flight back to the earth. What characterizes this time? This is the time for the \( z \) value to return to \( z = 0 \). To obtain this we take the expression for the \( z \) component of the motion as a function of time given in (5) and equate it to zero, solving for \( t \):

\[ (v_0 \sin \alpha)t - \frac{g}{2}t^2 = t \left( v_0 \sin \alpha - \frac{g}{2}t \right) = 0, \]

which gives:

\[ t = \frac{2v_0 \sin \alpha}{g}. \]

\( (t = 0 \) is also a solution. Why do we discount it?) Note that this is twice the value that it takes to reach the maximum height. Would you expect that?

Finally, we compute the range. This is the maximum distance that the projectile travels horizontally (i.e., in the \( y \) direction). To compute this we substitute the time that it takes to return back to earth into the expression for the \( y \) component given in (5):

\[ \text{Range} = (v_0 \cos \alpha) \left( \frac{2v_0 \sin \alpha}{g} \right) = \frac{2v_0^2 \sin \alpha \cos \alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g}. \]
Example. A particle $P$ of constant mass $m$ slides without rolling down a frictionless inclined plane of angle $\alpha$, see Fig. 3.

If the particle starts from rest at the top of the incline (at point $A$) find:

1. the acceleration,
2. the velocity,
3. the distance traveled after time $t$.

Since there is no friction the only forces acting on the particle are the weight $W = -mgk$ and the reaction force of the incline which is given by the normal force $N$.

Let $e_1$ and $e_2$ be unit vectors parallel and perpendicular to the incline, respectively. We let $s$ denote the magnitude of the displacement of the particle measured from the top of the incline (point $A$). Now we can write down Newton’s second law for this system:

$$m \frac{d^2 (se_1)}{dt^2} = W + N = mg \sin \alpha e_1,$$

or

$$\frac{d^2 s}{dt^2} = g \sin \alpha,$$

which gives the acceleration.

Next we compute the speed. Since $\frac{ds}{dt} = v$ we have:

$$\frac{dv}{dt} = g \sin \alpha,$$

or

$$v = (g \sin \alpha) t + c_1$$

Using the initial condition $v = 0$ at $t = 0$, we obtain $c_1 = 0$. Hence, the speed at any time $t$ is given by:

$$v = (g \sin \alpha) t.$$

The velocity at any time $t$ is given by:

$$v e_1 = (g \sin \alpha) t e_1.$$

Finally, we compute the distance traveled after time $t$. Since $v = \frac{ds}{dt}$ we have:

$$\frac{ds}{dt} = (g \sin \alpha) t \quad \text{or} \quad s = \frac{1}{2} (g \sin \alpha) t^2 + c_2.$$

Using the initial condition $s = 0$ at $t = 0$ we find that $c_2 = 0$, and therefore

$$s = \frac{1}{2} (g \sin \alpha) t^2.$$
**Example.** A particle, denoted by $P$, sits at the top (point $A$ in the figure) of a frictionless fixed sphere of radius $b$, see Fig. 4.

![Figure 4:](image)

The particle is displaced slightly so that it slides (without rolling) down the sphere. Compute:

1. the position of the particle as it leaves the sphere,
2. the speed of the particle at the instant it leaves the sphere.

The particle slides along a circle which, as indicated in the figure, we have taken to lie in the $x − y$ plane. The forces acting on the particle are its weight, $W = -mg$, and the reaction force $N$ of the sphere on the particle normal to the sphere.

We let the position of the particle on the circle be measured by the angle $\theta$, and we let $r_1$ and $\theta_1$ denote unit vectors in the radial direction and the direction of increasing $\theta$, respectively (see Fig. 4).

For this problem it is most natural to use a (moving) coordinate system tangential and normal to the sphere since the question of interest is how far the particle has moved along the sphere when it leaves the sphere.

Therefore, we want to express Newton’s equations in this coordinate system. We begin by resolving the forces into components along these directions. First, as a preliminary calculation, note from Fig. 4, that we have:

\[
\begin{align*}
\mathbf{j} \cdot r_1 &= |\mathbf{j}| |r_1| \sin \theta = \sin \theta, \\
\mathbf{j} \cdot \theta_1 &= |\mathbf{j}| |\theta_1| \cos \theta = \cos \theta,
\end{align*}
\]

We use these relations to resolve the force due to gravity on the particle in the $r_1$ and $\theta_1$ directions. This is given by:

\[
\mathbf{W} = (\mathbf{W} \cdot r_1)r_1 + (\mathbf{W} \cdot \theta_1)\theta_1,
\]

\[
= (-mg \mathbf{j} \cdot r_1)r_1 + (-mg \mathbf{j} \cdot \theta_1)\theta_1,
\]

\[
= -mg \sin \theta r_1 - mg \cos \theta \theta_1.
\]

The normal reaction force clearly acts in radial direction (since this is normal to the sphere):
\[ N = Nr_1. \]

Hence, Newton’s laws become:

\[
F = ma = m \left( (\ddot{r} - r\ddot{\theta}^2) r_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \theta_1 \right),
\]

\[ = W + N,
\]

\[ = (N - mg \sin \theta) r_1 - mg \cos \theta \theta_1,
\]

where the expression for the acceleration is given by the second derivative of the position vector expressed in polar coordinates which was discussed in some detail in the notes from Week 15.

Alternately, we can write the equations separately in components as follows:

\[ m(\ddot{r} - r\ddot{\theta}^2) = N - mg \sin \theta, \quad m(\dddot{r} + 2\dot{r}\dot{\theta}) = -mg \cos \theta. \tag{6} \]

But in this problem, the particle is constrained to move along the sphere (until it leaves, but we won’t be concerned with the motion after this happens), so we have \( r = b \), and therefore \( \dot{r} = \ddot{r} = 0 \). Therefore, the equations of motion simplify to:

\[ -mb \dot{\theta}^2 = N - mg \sin \theta, \quad b\ddot{\theta} = -g \cos \theta. \tag{7} \]

Now that we have written down the equations of motion we have to think what we have been asked to compute. The first thing we want to know is at what position (i.e., value of \( \theta \)) will the particle leave the sphere? We must ask ourselves what would characterize this event (in terms of the applied forces)? The particle leaves the sphere when the normal reaction force is zero (think about this). Therefore, we would like to find an expression for the normal force as a function of \( \theta \), then equate the expression to zero and solve for \( \theta \).

How do we find an expression for the normal force? We have to go to the equations of motion (7), and be a bit clever.

If we multiply the second equation in (7) by \( \dot{\theta} \), we see that the result can be written as:

\[ b \frac{d}{dt} \left( \frac{\dot{\theta}^2}{2} \right) = -g \frac{d}{dt}(\sin \theta). \]

Integrating this equation gives:

\[ b \frac{\dot{\theta}^2}{2} = -g \sin \theta + c. \]

Now at the top of the sphere, \( \theta = \frac{\pi}{2} \), we have \( \dot{\theta} = 0 \). Therefore \( c = g \), and we have:

\[ b \dot{\theta}^2 = 2g(1 - \sin \theta). \tag{8} \]

Substituting this expression into the first equation of (7) gives:

\[ N = mg(3 \sin \theta - 2), \]

which is our sought after expression for the normal force as a function of \( \theta \).

Equating this expression to zero gives:

\[ \sin \theta = \frac{2}{3}, \quad \text{or} \quad \theta = \sin^{-1} \frac{2}{3}. \tag{9} \]

Finally, we want to find the speed of the particle at the moment it leaves the sphere. The angular speed of the particle while it is on the sphere is given by \( |\dot{\theta}| \).

If we substitute (9) into (8) we obtain:
\[
\dot{\theta}^2 = \frac{2g}{3b}.
\]

So the angular speed of the particle when it leaves the sphere is given by:

\[|\dot{\theta}| = \sqrt{\frac{2g}{3b}}.
\]

Therefore speed of the particle when it leaves the sphere is given by:

\[|\dot{\phi}| = \sqrt{\frac{2}{3}gb}.
\]

Key point: "How do you solve mechanics problems?"

Initially, this might seem to be a curious statement since the point of the lectures is to provide you with the tools for solving mechanics problems. They do provide you with the tools. However, in the first year many students find the problems in mechanics more difficult than those in, say, calculus. There is probably some truth to this, so we should look at this in more detail.

First, in calculus many of the problems that you are set involve performing a specific computation that should be clear from the statement of the problem. For example, you might be given a scalar valued function of two variables and asked to compute its (first) partial derivatives. Hopefully, you understand that the solution of that problem is a straightforward calculation. Another example involves integration of a function of one variable. You may be given a function and asked to integrate it between two limits. What you are asked to do should be straightforward. However, some insight (coming from experience) is required in order to know what technique to employ (e.g. partial fractions, integration by parts, etc.) to successfully integrate the given function. Knowing how to start with a mechanics problem is essential to answering the questions related to the problem that you are asked. Let's consider this in the context of the first two examples in the Week 18 Lecture Notes.

The first example concerns the motion of a projectile. You are given a one sentence description of the problem, and then asked to determine 5 quantities associated with the motion of the projectile. The natural thing to do is to focus on the 5 quantities that you are asked to determine. But this is not where you start. Instead, you should first focus on writing down Newton's equations, i.e. the second order differential equation that is a consequence of Newton's second law of motion. This involves determining the forces acting on the particle (at the moment we are concerning with one particle of constant mass \(m\)). Once this is done, you can write down Newton's equations as a vector equation (the mass multiplied by the second derivative of the position vector is equal to the vector sum of all of the forces acting on the particle). Now the motion tends to be described in terms of a particular coordinate system (e.g. how high does the particle go? how far does it go? where is it along the inclined plane?). In order to do this you will need to choose a set of coordinates that are appropriate to the problem in which to represent Newton's equations, i.e. represent the vector quantities in Newton's equations in components with respect to the chosen coordinates. Once you have carried out these steps (determine the forces acting on the particle, write down Newton's equations, choose an appropriate set of coordinates in which to represent Newton's equations) then you will be ready to answer the questions concerning the motion of the particle that you are asked. However, note that when determining Newton's equations (i.e. the nature of the vector sum of forces acting on the particle) you may have to use Newton's first and third laws of motion. See how this arises in the examples in this week's notes.