

REVIEW ARTICLE

An optimisation approach for analysing nonlinear stability with transition to turbulence in fluids as an exemplar

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Abstract.

This article introduces and reviews recent work using a simple optimisation technique for analysing the nonlinear stability of a state in a dynamical system. The technique can be used to identify the most efficient way to disturb a system such that it transits from one stable state to another. The key idea is introduced within the framework of a finite-dimensional set of ordinary differential equations (ODEs) and then illustrated for a very simple system of 2 ODEs which possesses bistability. Then the transition to turbulence problem in fluid mechanics is used to show how the technique can be formulated for a spatially-extended system described by a set of partial differential equations (the well-known Navier-Stokes equations). Within that context, the optimisation technique bridges the gap between (linear) optimal perturbation theory and the (nonlinear) dynamical systems approach to fluid flows. The fact that the technique has now been recently shown to work in this very high dimensional setting augurs well for its utility in other physical systems.

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1. Introduction

Many physical systems possess a multiplicity of stable states so that more than one solution or system configuration can be found at long times. In such situations, a key issue is usually maintaining the system in a desired state against ambient noise or switching the system from one (undesirable) state to another (preferred) state in an efficient, robust way: e.g. in liquid crystal displays [76], power grids [84], arrays of coupled lasers [41], turbulent fluid flows [51] and even in the human brain [3]. Either objective involves detailed knowledge of a state's basin of attraction, defined as the set of all initial conditions of the system whose long time behaviour is to converge to that state. Initial conditions located just outside the basin boundary indicate how the system can be efficiently disturbed to trigger a new stable state. Knowledge of how the basin boundary of a state moves (in phase space) when the system is manipulated (e.g. by modifying the boundary conditions) opens up the possibility of enhancing the nonlinear stability of that state to finite amplitude disturbances. However, locating a basin boundary is a fully nonlinear (nonlocal) problem so that the traditional tools of linearising the system around the state or even weakly nonlinear analysis provide no traction. Existing fully nonlinear approaches - solving the governing equations while searching for the finite-amplitude disturbances to just knock the system out of one state into another, or mapping out the stable and unstable manifolds of nearby solutions in phase space to identify the basin boundary - are impractical for all but the smallest systems.

Recently, a new, very general, fully nonlinear optimisation technique has emerged as a viable way to make progress. The underlying idea is relatively simple and, perhaps because of this, seems to have been formulated independently in (at least) three different parts of the scientific literature over the last decade (transitional shear flows [18, 86, 104] - see §3, oceanography [89] - see §4.1 and thermoacoustics [69] - see §4.2). The key advance, however, has come in the last few years when the feasibility of the approach has been demonstrated for the 3-dimensional Navier-Stokes equations discretised by a large number ($O(10^5-10^6)$) of degrees of freedom [18, 19, 21, 37, 86, 104, 105, 108]. This suggests that other partial differential equation systems could be usefully analysed with this approach.

This article (which is an updated and extended version of [74]) is an attempt to provide a simple introduction to the idea and to review the progress made so far. As should become clear, the approach is still developing but there is already enough evidence garnered to indicate that it adds something quite new to a theoretician's toolbox. In fluid mechanics, the last two decades have seen a huge amount of work looking at flow transition either from the linear transient growth perspective (also called 'non-modal analysis' or 'optimal perturbation theory' [54, 113, 117, 116]) or more recently in terms of exact solutions and their manifolds (a dynamical systems approach [73, 39, 72]). The optimisation technique discussed here bridges the well-known 'amplitude' gap between these two viewpoints by extending the (infinitesimal

amplitude) transient growth optimal of the former approach into finite amplitudes and ultimately up to where the basin boundary is crossed (the closest stable manifold of a nearby exact solution).

The plan of this article is as follows. In section 2, the central idea of the optimisation approach is introduced in the context of a finite-dimensional dynamical system. A simple system of 2 ordinary differential equations (ODEs) is then used to: a) illustrate the results of a calculation, and b) highlight some important ingredients which make the technique work. In section 3, the discussion is moved onto fluid mechanics and the Navier-Stokes equations - a time-dependent and 3-space dimension nonlinear partial differential equation (PDE). The first attempted use of the optimisation technique was in the classical problem of flow through a pipe [104] and so this is used here as the context to explain the inner workings of the approach. Appendices A and B provide simple supporting illustrations of a linear transient growth calculation for a system of 2 ODEs and how including nonlinearity can allow different growth mechanisms to work together to produce far greater overall growth. Section 4 reviews some of the literature which preceded the successful application to the Navier-Stokes equations as well as discussing further work now building on it. Finally section 5 provides a summary and surveys what the future may hold.

2. Basic Idea

To explain the optimisation technique, consider a finite-dimensional dynamical system

$$\frac{d\mathbf{X}}{dt} = \mathbf{f}(\mathbf{X}; \mu) \quad (1)$$

where $\mathbf{X} = \mathbf{X}(t) \in \mathbb{R}^N$ and μ is a parameter of the system. Let \mathbf{X}_0 be a local steady attractor of interest and $\mathbf{x} := \mathbf{X} - \mathbf{X}_0$ be the perturbation away from this attractor. Then write the evolution equation for \mathbf{x} as

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}; \mathbf{X}_0, \mu) \quad (2)$$

and define a norm $\|\mathbf{x}(t)\|$ to measure the distance of $\mathbf{X}(t)$ from \mathbf{X}_0 . The approach is to find the maximum distance after some time $t = T$, $\|\mathbf{x}(T)\|$, over all perturbations which start the same finite distance

$$\|\mathbf{x}(0)\| = d \quad (3)$$

away at time $t = 0$ and evolve under (2). For the Euclidean norm, $\|\mathbf{x}\|_2 := \sqrt{\sum_{n=1}^N x_n^2}$, this can be formulated particularly easily as maximising the Lagrangian

$$\mathcal{L} = \mathcal{L}(\mathbf{x}, \boldsymbol{\nu}, \lambda; \mathbf{X}_0, d, T) := \|\mathbf{x}(T)\|_2^2 + \int_0^T \boldsymbol{\nu} \cdot \left(\frac{d\mathbf{x}}{dt} - \mathbf{F} \right) dt + \lambda (\|\mathbf{x}(0)\|_2^2 - d^2) \quad (4)$$

with $\boldsymbol{\nu}(t)$ and λ acting as Lagrange multipliers to impose the dynamical constraint (2) and initial distance constraint respectively and ‘.’ is the usual scalar product. Maximal

values of \mathcal{L} are identified by vanishing first variations with respect to each of $\mathbf{x}(t)$, $\boldsymbol{\nu}(t)$ and λ (\mathbf{X}_0 , d and T are fixed). The first variation of \mathcal{L} with respect to $\mathbf{x}(t)$ is

$$\begin{aligned} \delta\mathcal{L} &:= \lim_{\epsilon \rightarrow 0} \frac{\mathcal{L}(\mathbf{x} + \epsilon\delta\mathbf{x}, \boldsymbol{\nu}, \lambda; \mathbf{X}_0, d, T) - \mathcal{L}(\mathbf{x}, \boldsymbol{\nu}, \lambda; \mathbf{X}_0, d, T)}{\epsilon} \\ &= [2\mathbf{x}(T) + \boldsymbol{\nu}(T)] \cdot \delta\mathbf{x}(T) - \int_0^T \left[\frac{d\boldsymbol{\nu}}{dt} + \boldsymbol{\nu} \cdot \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \right] \cdot \delta\mathbf{x} dt \\ &\quad + [2\lambda\mathbf{x}(0) - \boldsymbol{\nu}(0)] \cdot \delta\mathbf{x}(0) \end{aligned} \quad (5)$$

which only vanishes for all allowed variations $\delta\mathbf{x}(t)$ if

$$\frac{d\boldsymbol{\nu}}{dt} + \boldsymbol{\nu} \cdot \frac{\partial \mathbf{F}}{\partial \mathbf{x}} = \mathbf{0} \quad \text{over } t \in (0, T) \quad (6)$$

and

$$2\lambda\mathbf{x}(0) - \boldsymbol{\nu}(0) = 2\mathbf{x}(T) + \boldsymbol{\nu}(T) = \mathbf{0}. \quad (7)$$

Stationarity of \mathcal{L} with respect to the Lagrange multipliers $\boldsymbol{\nu}$ and λ by construction imposes the evolution equation (2) and the initial distance constraint (3) respectively. Maximising \mathcal{L} is then a problem of simultaneously satisfying (2), (3), (6) and (7). This is in general a nonlinear system that needs to be solved iteratively. The solution technique starts with an initial guess $\mathbf{x}(0)$ which is integrated forward in time using (2) to produce $\mathbf{x}(T)$. This initializes (via (7)) the *backward* integration of the ‘dual’ or ‘adjoint’ dynamical equation

$$\frac{d\boldsymbol{\nu}}{dt} = -\boldsymbol{\nu} \cdot \frac{\delta \mathbf{F}}{\delta \mathbf{x}} \quad (8)$$

(with $\delta \mathbf{F} / \delta \mathbf{x}$ a matrix and generally dependent on $\mathbf{x}(t)$) from $t = T$ back to $t = 0$ to generate $\boldsymbol{\nu}(0)$. At this point, only two conditions remain to be satisfied. The Fréchet derivative $\delta\mathcal{L}/\delta\mathbf{x}(0) := 2\lambda\mathbf{x}(0) - \boldsymbol{\nu}(0)$ will *not* in general vanish so the strategy is to move $\mathbf{x}(0)$ (subject to the initial distance constraint) until it does. By choosing to ‘ascend’ (moving $\mathbf{x}(0)$ in the direction of $\delta\mathcal{L}/\delta\mathbf{x}(0)$) a maximum in \mathcal{L} is sought. The value of λ is simultaneously specified by ensuring that $\|\mathbf{x}(0)\|_2 = d$ continues to hold during this adjustment in $\mathbf{x}(0)$. This procedure is repeated until $\|\delta\mathcal{L}/\delta\mathbf{x}(0)\|_2$ is sufficiently small to indicate a maximum. Theoretically, when $T \rightarrow \infty$, the global maximum

$$\max_{\mathbf{x}(0)} \mathcal{L} = \begin{cases} 0 & d < d_c, \\ \|\mathbf{X}_s - \mathbf{X}_0\|_2^2 & d = d_c, \\ \|\mathbf{X}_1 - \mathbf{X}_0\|_2^2 & d > d_c \end{cases} \quad (9)$$

where \mathbf{X}_1 is another stable state of the system, \mathbf{X}_s is a saddle embedded in the basin boundary with its only unstable manifold perpendicular to the boundary and in general $\|\mathbf{X}_s - \mathbf{X}_0\|_2 \neq \|\mathbf{X}_1 - \mathbf{X}_0\|_2$ (for simplicity \mathbf{X}_s and \mathbf{X}_1 are assumed steady). This is just the statement that depending on whether the initial perturbation is strictly in the basin of attraction of \mathbf{X}_0 , on the basin boundary or outside the basin (and in the basin of attraction of \mathbf{X}_1), the endstate is \mathbf{X}_0 , \mathbf{X}_s or \mathbf{X}_1 respectively for large times. The jump in the endstate distance is then discontinuous once d reaches d_c (*a priori* unknown) which signals that the basin boundary has been reached. The optimal disturbance $\mathbf{x}(0)$

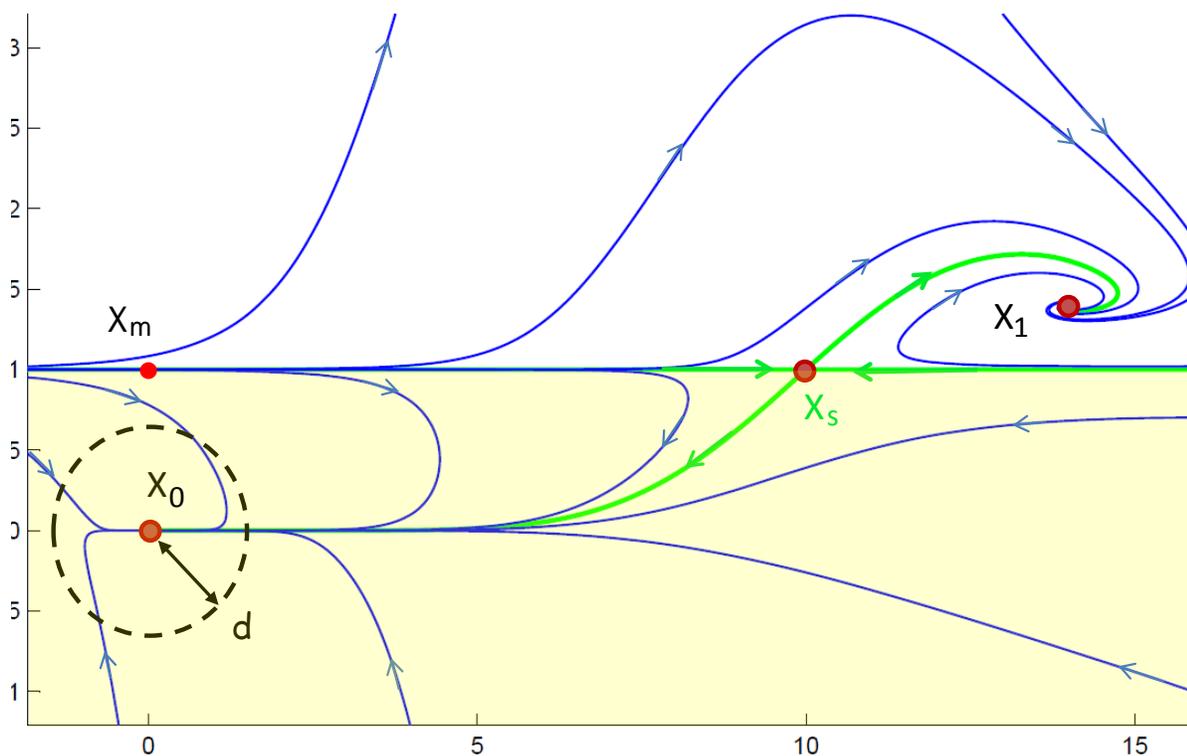


Figure 1. Phase portrait for the system defined by (10). The stable and unstable manifolds of \mathbf{X}_s are drawn in thick green. A circle of radius $\|\mathbf{x}(0)\|_2 = d$ about \mathbf{X}_0 is shown as a dashed line. As the radius increases, the circle first touches the basin boundary at $\mathbf{X}_m = (0, 1)$ - the *minimal seed* since it is arbitrarily close to a state which will trigger the new asymptotic state \mathbf{X}_1 when the system is in state \mathbf{X}_0 .

at $d = d_c$ has been christened the ‘minimal seed’ [104, 105] since it is arbitrarily close to a disturbance which will trigger the transition from \mathbf{X}_0 to \mathbf{X}_1 for $d = d_c^+$. Put another way, the minimal seed, renormalised by a factor $1 + \epsilon$ where ϵ is vanishingly small but non-zero, represents the easiest/most efficient way to leave the basin of attraction of \mathbf{X}_0 . Practically, this behaviour can be mimicked by choosing T large enough that trajectories starting sufficiently close to the basin boundary but either side of it can be observed to diverge. The minimal seed is then approximated by the first initial condition as d is increased which is observed to have escaped the basin of attraction of \mathbf{X}_0 .

2.1. A simple example

We show how this technique works for the simple two-dimensional system

$$\frac{d\mathbf{X}}{dt} = \begin{bmatrix} -X_1 + 10X_2 \\ X_2(10e^{-X_1^2/100} - X_2)(X_2 - 1) \end{bmatrix} \quad (10)$$

where $\mathbf{X} = (X_1, X_2)$ which has two local attractors at $\mathbf{X}_0 = (0, 0)$ and $\mathbf{X}_1 = (14.017, 1.4017)$. The basin boundary is simply $X_2 = 1$ which is the stable manifold of the saddle point at $\mathbf{X}_s = (10, 1)$: see Figure 1. All initial conditions with $X_2 < 1$ are

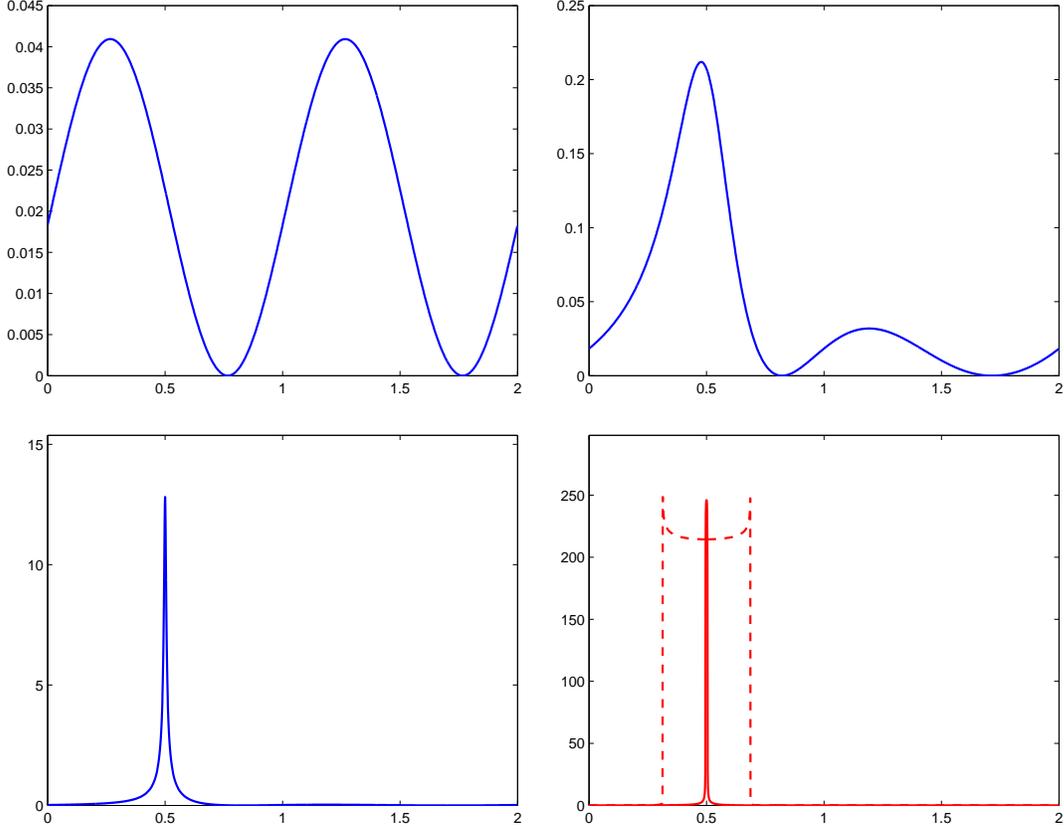


Figure 2. Top row: Gain $G := \|\mathbf{x}(t)\|_2^2/d^2$ versus θ for $T = 2$ and $d = 10^{-4}$ (top left) or 0.9 (top right). Bottom row: $\|\mathbf{x}(T)\|_2^2$ versus θ for $T = 2$ and $d = 0.9999$ (bottom left) or 1.0001 (solid line, bottom right) and 1.2 (dashed line, bottom right). Notice the sudden jump in $\|\mathbf{x}(T)\|_2^2$ when d increases from 0.9999 to 1.0001 signalling that the basin boundary of \mathbf{X}_0 has been crossed ($\|\mathbf{x}(T)\|_2^2$ is simply $100(1 - e^{-T})^2 + 1$ for $d = d_c$).

attracted to \mathbf{X}_0 , those with $X_2 > 1$ to \mathbf{X}_1 , and those with $X_2 = 1$ to \mathbf{X}_s . Focussing on the ‘unexcited’ state \mathbf{X}_0 , $\mathbf{x} := \mathbf{X} - \mathbf{X}_0$ and using the Euclidean norm again for simplicity, it is clear that drawing a circle in the (x_1, x_2) plane centred on the origin and of increasing radius d , the basin boundary for \mathbf{X}_0 will be first touched at $d = d_c = 1$ where $(x_1, x_2) = (0, 1)$: this is the *minimal seed* for this system. As d increases beyond d_c , the basin boundary is increasingly punctured so that ever more initial conditions are outside of the basin of attraction of \mathbf{X}_0 .

This observation can be deduced by iteratively solving the optimisation problem described above through integrating (10) forwards (since here $\mathbf{x} = \mathbf{X}$ as $\mathbf{X}_0 = \mathbf{0}$) to find $\mathbf{x}(t)$ and the dual dynamical equation

$$\frac{d\boldsymbol{\nu}}{dt} = \begin{bmatrix} \nu_1 + \frac{1}{5}x_1\nu_2(x_2^2 - x_2)e^{-x_1^2/100} \\ -10\nu_1 + \nu_2(3x_2^2 - 2x_2) - 10\nu_2(2x_2 - 1)e^{-x_1^2/100} \end{bmatrix} \quad (11)$$

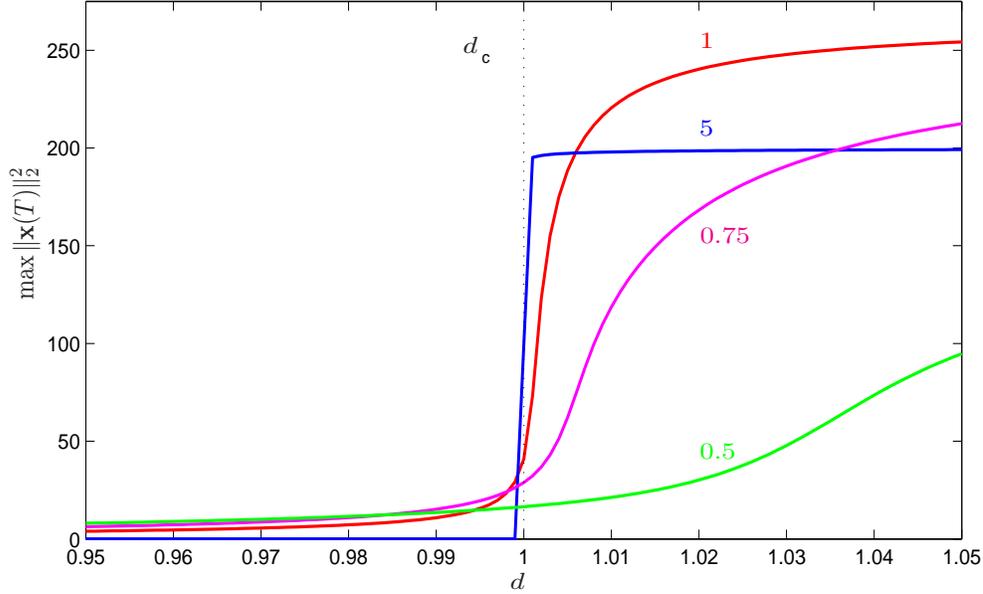


Figure 3. As T (labelled on each curve) increases, the signature of the transition at d_c becomes increasingly clear. Note only for $T \rightarrow \infty$ is the final plateau $\|\mathbf{x}(T)\|_2^2 = \|\mathbf{X}_1 - \mathbf{X}_0\|_2^2 \approx 198.5$ as $\mathbf{x}(t)$ ‘overshoots’: see Figure 1.

for $\boldsymbol{\nu}(t)$ backwards (note that this backwards-in-time integration requires knowledge of $\mathbf{x}(t)$ across $[0, T]$). Figure 2 shows how $\|\mathbf{x}(T)\|_2^2$ varies over all permitted initial conditions parameterised by the angle $\theta \in [0, 2)$ where

$$\mathbf{x}(0) = (d \cos \pi\theta, d \sin \pi\theta) \quad (12)$$

for selected values of d and $T = 2$ (note the gain $G := \|\mathbf{x}(T)\|_2^2/d^2$ is plotted for the top row of subplots since d has such different values). The small value $d = 10^{-4}$ is used to reproduce the linear result obtainable via standard matrix manipulations (see Appendix A) which possesses the $\mathbf{x}(0) \rightarrow -\mathbf{x}(0)$ ($\theta \rightarrow \theta \pm 1$) symmetry. As d is increased, this symmetry is quickly destroyed as nonlinearity becomes important to indicate a unique optimal value of θ approaching 0.5 as $d \rightarrow d_c$. Once d reaches d_c , $\|\mathbf{x}(T)\|_2^2$ jumps in value as the system explores the basin boundary and then (for $d > d_c$) the basin of attraction of \mathbf{X}_1 . Three key points highlighted by this simple example are as follows.

- (i) The importance of T in determining d_c to a required accuracy. Figure 3 shows that d_c is increasingly well identified as the value of T increases or if T is too small, the jump in $\|\mathbf{x}(T)\|_2^2$ at $d = d_c$ is smeared out and so difficult to locate. On a practical level, T should not be chosen too large as the optimisation procedure becomes increasingly sensitive to changes in the starting state.
- (ii) The minimal seed is *not* related to the linear optimal. For ‘large’ T (≥ 2), the minimal seed has $\theta^* = 0.5$ (90°) whereas the linear optimal ($d \rightarrow 0$) has $\theta^* \approx 0.2667$ (48.0°). This is not surprising since phase space immediately around \mathbf{X}_0 is generally

unrelated to the structure of the basin boundary a finite distance away. (The one caveat to this is if the nonlinear terms had been chosen to be energy-preserving - e.g. the model system studied in [30]. In this case, the dynamics are so tightly constrained in 2 dimensions that the minimal seed has to be the same as the linear optimal: see Appendix B for further discussion.)

- (iii) The Euclidean norm works well as there is little growth for trajectories with $x_2 < 1$, whereas trajectories in $x_2 > 1$ overshoot \mathbf{X}_1 . If this were not the case, the functional to optimise - the objective functional - could be redesigned to signal the arrival in a new basin. Here for example, the value of $x_2(T)$ would be more appropriate with the Euclidean norm still used to constrain the competitor set of initial perturbations, i.e. the measure used as the objective functional and the norm constraining the initial condition do not need to be the same.

3. The Transition to Turbulence problem in Fluid Mechanics

We now move the discussion to fluid mechanics and a system described by a set of PDEs. The breakdown of laminar shear flows has been a central problem in fluid mechanics since the inception of the subject and fascinated many generations of scientists (e.g. Rayleigh [110], Kelvin [127], Reynolds [114, 115], Orr [99], Sommerfeld [124], Noether [97], Taylor [125], Heisenberg [58], Landau [78], Hopf [61] and see the textbooks [17, 33, 32, 117]). Beyond a certain value of the driving rate (measured by a non-dimensional grouping called the Reynolds number Re and generated by either imposed boundary motion, flow rate or pressure gradient), unidirectional shear flows (e.g. flow through a straight pipe) typically exhibit *bistability* where a linearly-stable simple laminar state coexists with a spatially- and temporally-complicated turbulent state. The so-called ‘transition’ problem consists in understanding the physical processes by which ambient noise (present in any real flow) can trigger the observed transition of the flow from the laminar to the turbulent state. The fact that the laminar state is linearly stable means that the transition process is inherently nonlinear and even now still largely unexplained. The transition problem is not only a fascinating mathematical exercise in PDE theory, but the answers are crucial for informing attempts to inhibit (e.g. in the aircraft industry) or enhance (e.g. crucial for mixing processes) the phenomenon in practical applications.

In the last two decades or so, two complementary approaches to the transition problem have proved popular. The first - variously labelled ‘transient growth’, ‘nonmodal stability theory’ or ‘optimal perturbation theory’ - is a *linear* theory explaining how, due to the non-normality of the linearised evolution operator[†] about the laminar state, infinitesimal disturbances can experience large but transient magnification of their energy despite the laminar state being asymptotically stable (all the eigenvalues indicate exponential decay with time). This idea has a long history starting with Kelvin

[†] An operator/matrix L is non-normal if it does not commute with its adjoint/transpose L^\dagger i.e. $LL^\dagger \neq L^\dagger L$.

[127] and Orr [99] but has really only been systematically explored from the late 1980s onwards: see [12, 43, 57, 15, 130, 111], the reviews [54, 113, 116, 118] and the books [117, 131]. The theory works well for interpreting finite time behaviour such as the flow response to noise immediately downstream of a pipe expansion [16] but actually says nothing about asymptotically long time behaviour. Initially, it was suggested that the presence of large transient growth could ‘elevate’ infinitesimal disturbances into the nonlinear regime where they could then become sustained (e.g. [4, 5]). While this picture, which emphasizes linear effects over nonlinearity, must be generally correct (as only linear effects can give rise to energy growth), no *quantitative* predictions of transition amplitudes can emerge without a fully nonlinear theory [133, 30].

The second approach to the transition problem, on the other hand, is fully nonlinear. This views the flow as a dynamical system and the flow state as an evolving trajectory in a phase space populated by various invariant sets (exact solutions) and their stable and unstable manifolds [38, 73, 39, 53, 29, 72]. From this perspective, ‘transition’ occurs when noise or a disturbance simply nudges the flow out of the basin of attraction of the laminar state. Making this observation more predictive has, however, been extremely hard due to the difficulty of mapping out the basin boundary. Some progress has been made shadowing the basin boundary (or the more general concept of an ‘edge’, which includes transient turbulence) *forward* in time using an ‘edge tracking’ technique. This finds ‘edge states’ which are saddles in the full phase space but attractors on the basin boundary [63, 123, 119] (e.g. \mathbf{X}_s in Figure 1). However, these are invariably more distant (energetically) from the laminar state [105] than other parts of the basin boundary and therefore less obviously relevant for choosing an initial transition-triggering disturbance. What is really needed is a technique to track the basin boundary *backwards* in time to reach regions where it is close to the laminar state.

The gap between these two approaches is therefore one of perspective: the optimal perturbation theory explains how infinitesimal disturbances can grow temporarily *within* the basin boundary whereas the dynamical systems approach focusses on what exists *beyond* the basin boundary. The new optimisation technique discussed in this article naturally bridges this gap by being able to examine how the optimal perturbation deviates from the linear optimal as the amplitude of the starting perturbation is increased until the basin boundary is reached. The original thinking behind the studies [104, 105], however, was a little different being focussed on tracing the basin boundary or edge ‘backwards’ in time by posing an optimisation problem. If the edge state found by edge tracking is unique then for all starting states *on* the basin boundary, the initial state which subsequently experiences the largest energy growth over asymptotically large times (so that all trajectories reach the edge state) *will* be the minimal seed [105]. While this optimisation problem itself is intractable because the basin boundary is unknown, it does suggest the tractable problem of finding the largest energy growth over all starting states of a given initial energy E_0 . At precisely $E_0 = E_c$, where the basin boundary or edge touches the energy hypersurface at one velocity state, this optimization problem considers the growth of this state (the minimal seed) against the energy growth of all the

other initial conditions below the edge. Given that these latter initial conditions lead to flows that grow initially but ultimately relax back to the basic state, the minimal seed will remain the optimal initial condition for the revised optimisation problem for large enough T [105]. The simple choice of the kinetic energy of the disturbance as both the objective functional *and* the constraining norm meant that the vanishing energy limit $E_0 \rightarrow 0$ lead back to the familiar linear optimal perturbation calculation. It is now clear other choices could have been made for the objective functional providing it takes on heightened values for the turbulent state (e.g energy dissipation [86, 37] and [47, 48] for work considering different norms).

Before giving an example of the optimisation technique at work in fluid mechanics, we note that there have been many previous efforts to build upon (linear) optimal perturbation theory to design better ways to trigger transition. For example (since space prevents an exhaustive listing), by considering secondary transient growth on top of a linear optimal (e.g. [122, 60, 27]), or by probing its finite amplitude stability (e.g. [28]), or by showing that the linear optimals can become linearly unstable at sufficient amplitude (e.g. [139, 112]). Finite amplitude initial disturbances have also been designed from a very small set of physically-motivated ‘basis’ states (e.g. [132, 36]) and optimal deformations of the *base flow* found so as to create linear instability [13, 10, 52, 11]. All this work either focusses on one possible aspect of the transition process or makes some assumption about disturbances able to trigger transition efficiently. In contrast, no such assumptions are made in what follows.

3.1. The optimisation technique in pipe flow

We now show the optimisation technique in action for the problem of incompressible fluid flow through a cylindrical pipe. This is a classical problem in fluid mechanics studied famously by Reynolds [114, 115] during which he first wrote down the non-dimensional grouping Re now bearing his name. The flow can either be driven by imposing a constant pressure drop across the pipe or by imposing a constant mass flux through the pipe (e.g. [96]). We choose the former driving here as the formulation is slightly simpler (the latter is treated in [104, 105]): the two cases are equivalent for $L \rightarrow \infty$ and localised disturbances. The set up is an incompressible fluid of constant density ρ and kinematic viscosity ν flowing in a circular pipe of radius s_0 under the action of a constant pressure drop imposed across the pipe of

$$\Delta p^* = -\frac{4\rho\nu W}{s_0}L \quad (13)$$

(where the pipe is L radii long) and the (basic) Hagen-Poiseuille solution to the Navier-Stokes equations is

$$\mathbf{u}_{\text{lam}}^* := W\left(1 - \frac{s^{*2}}{s_0^2}\right)\hat{\mathbf{z}}^*, \quad p_{\text{lam}}^* := -\frac{4\rho\nu W}{s_0^2}z^* \quad (14)$$

in the usual cylindrical coordinates (s^*, ϕ, z^*) aligned with the pipe axis. Non-dimensionalizing the system using the Hagen-Poiseuille centreline speed W (e.g. $\mathbf{u}_{\text{lam}} :=$

$\mathbf{u}_{\text{lam}}^*/W$) and the pipe radius s_0 (so $s := s^*/s_0$) gives the Navier-Stokes equations

$$\frac{\partial \mathbf{u}_{\text{tot}}}{\partial t} + \mathbf{u}_{\text{tot}} \cdot \nabla \mathbf{u}_{\text{tot}} + \nabla p_{\text{tot}} = \frac{1}{Re} \nabla^2 \mathbf{u}_{\text{tot}} \quad (15)$$

where $Re := s_0 W / \nu$ is the Reynolds number and $p_{\text{tot}} := p + p_{\text{lam}}$ ($p_{\text{lam}} := -4z/Re$) is the total pressure with its deviation, p , from the laminar pressure field periodic across the length of the pipe to maintain the constant pressure drop: i.e. $p(s, \phi, z, t) = p(s, \phi, z + L, t)$. The velocity boundary conditions are

$$\mathbf{u}_{\text{tot}}(1, \phi, z, t) = \mathbf{0} \quad \& \quad \mathbf{u}_{\text{tot}}(s, \phi, z, t) = \mathbf{u}_{\text{tot}}(s, \phi, z + L, t), \quad (16)$$

the first being no-slip on the pipe wall at $s = 1$ and the second periodicity across the pipe length. We consider the energy growth (our ‘distance’ measure) of a finite-amplitude disturbance

$$\mathbf{u} := \mathbf{u}_{\text{tot}} - \mathbf{u}_{\text{lam}} \quad (17)$$

to the laminar profile $\mathbf{u}_{\text{lam}} = (1 - s^2)\hat{\mathbf{z}}$ by defining the Lagrangian

$$\begin{aligned} \mathcal{L} &= \mathcal{L}(\mathbf{u}, p, \lambda, \boldsymbol{\nu}, \pi; T, E_0) := \left\langle \frac{1}{2} |\mathbf{u}(\mathbf{x}, T)|^2 \right\rangle + \lambda \left\{ \left\langle \frac{1}{2} |\mathbf{u}(\mathbf{x}, 0)|^2 \right\rangle - E_0 \right\} \\ &+ \int_0^T \left\langle \boldsymbol{\nu}(\mathbf{x}, t) \cdot \left\{ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}_{\text{lam}} \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}_{\text{lam}} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right. \right. \\ &\quad \left. \left. + \nabla p - \frac{1}{Re} \nabla^2 \mathbf{u} \right\} \right\rangle dt + \int_0^T \langle \pi(\mathbf{x}, t) \nabla \cdot \mathbf{u} \rangle dt. \end{aligned} \quad (18)$$

where

$$\langle \dots \rangle = \int_0^L \int_0^{2\pi} \int_0^1 \dots s ds d\phi dz \quad (19)$$

and λ , $\boldsymbol{\nu}$ and π are Lagrangian multipliers imposing the constraints that the initial energy is fixed, that the Navier-Stokes equation holds over $t \in [0, T]$ and the flow is incompressible. Their corresponding Euler-Lagrange equations are respectively:

$$\left\langle \frac{1}{2} |\mathbf{u}(\mathbf{x}, 0)|^2 \right\rangle = E_0 \quad (20)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}_{\text{lam}} \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}_{\text{lam}} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \frac{1}{Re} \nabla^2 \mathbf{u} = \mathbf{0}, \quad (21)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (22)$$

The key additions to the well-known linear calculation acknowledging the fact that the disturbance is of finite amplitude are shown in red. The linearised problem is recovered in the limit of $E_0 \rightarrow 0$ whereupon the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ becomes vanishingly small relative to the other (linear) terms. On dropping this nonlinear term, the amplitude of the disturbance is then arbitrary for the purposes of the optimisation calculation and it is convenient to reset E_0 from vanishingly small to 1. In this case the maximum of \mathcal{L} is then precisely the maximum gain in energy over the period $[0, T]$.

The Euler-Lagrange equation for the pressure p is

$$\begin{aligned} 0 &= \int_0^T \left\langle \frac{\delta \mathcal{L}}{\delta p} \delta p \right\rangle dt = \int_0^T \langle (\boldsymbol{\nu} \cdot \nabla) \delta p \rangle dt \\ &= \int_0^T \langle \nabla \cdot (\boldsymbol{\nu} \delta p) \rangle dt - \int_0^T \langle \delta p (\nabla \cdot \boldsymbol{\nu}) \rangle dt. \end{aligned} \quad (23)$$

which vanishes if $\boldsymbol{\nu}$ satisfies natural boundary conditions (which turn out here to be the same as those for \mathbf{u}) and

$$\nabla \cdot \boldsymbol{\nu} = 0 \quad (24)$$

(note δp is periodic across the pipe to ensure the pressure drop across the pipe is constant). The variation in \mathcal{L} due to \mathbf{u} (with $\delta \mathbf{u}$ satisfying (16)) is

$$\begin{aligned} \delta \mathcal{L} &= \int_0^T \left\langle \frac{\delta \mathcal{L}}{\delta \mathbf{u}} \cdot \delta \mathbf{u} \right\rangle = \langle \mathbf{u}(\mathbf{x}, T) \cdot \delta \mathbf{u}(\mathbf{x}, T) \rangle + \lambda \langle \mathbf{u}(\mathbf{x}, 0) \cdot \delta \mathbf{u}(\mathbf{x}, 0) \rangle \\ &\quad + \int_0^T \left\langle \boldsymbol{\nu} \cdot \left\{ \frac{\partial \delta \mathbf{u}}{\partial t} + (\mathbf{u}_{\text{lam}} \cdot \nabla) \delta \mathbf{u} + (\delta \mathbf{u} \cdot \nabla) \mathbf{u}_{\text{lam}} \right. \right. \\ &\quad \left. \left. + \mathbf{u} \cdot \nabla \delta \mathbf{u} + \delta \mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{Re} \nabla^2 \delta \mathbf{u} \right\} \right\rangle dt + \int_0^T \langle \pi \nabla \cdot \delta \mathbf{u} \rangle dt. \end{aligned} \quad (25)$$

The first term in the second line of the above equation can be reexpressed as

$$\begin{aligned} \int_0^T \left\langle \boldsymbol{\nu} \cdot \frac{\partial \delta \mathbf{u}}{\partial t} \right\rangle dt &= \int_0^T \left\langle \frac{\partial}{\partial t} (\delta \mathbf{u} \cdot \boldsymbol{\nu}) \right\rangle dt - \int_0^T \left\langle \delta \mathbf{u} \cdot \frac{\partial \boldsymbol{\nu}}{\partial t} \right\rangle dt \\ &= \langle \delta \mathbf{u}(\mathbf{x}, T) \cdot \boldsymbol{\nu}(\mathbf{x}, T) - \delta \mathbf{u}(\mathbf{x}, 0) \cdot \boldsymbol{\nu}(\mathbf{x}, 0) \rangle - \int_0^T \left\langle \delta \mathbf{u} \cdot \frac{\partial \boldsymbol{\nu}}{\partial t} \right\rangle dt. \end{aligned} \quad (26)$$

Similarly, the second term is

$$\begin{aligned} \langle \boldsymbol{\nu} \cdot \{(\mathbf{u}_{\text{lam}} \cdot \nabla) \delta \mathbf{u}\} \rangle &= \langle \nabla \cdot ((\boldsymbol{\nu} \cdot \delta \mathbf{u}) \mathbf{u}_{\text{lam}}) - \delta \mathbf{u} \cdot \{(\mathbf{u}_{\text{lam}} \cdot \nabla) \boldsymbol{\nu}\} \rangle \\ &= - \langle \delta \mathbf{u} \cdot \{(\mathbf{u}_{\text{lam}} \cdot \nabla) \boldsymbol{\nu}\} \rangle. \end{aligned} \quad (27)$$

The third term is

$$\langle \boldsymbol{\nu} \cdot \{(\delta \mathbf{u} \cdot \nabla) \mathbf{u}_{\text{lam}}\} \rangle = \langle \delta \mathbf{u} \cdot \{ \boldsymbol{\nu} \cdot (\nabla \mathbf{u}_{\text{lam}})^T \} \rangle \quad (= \langle \delta u_i \nu_j \partial_i u_{\text{lam},j} \rangle). \quad (28)$$

The fourth and fifth terms are

$$\langle \boldsymbol{\nu} \cdot (\delta \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \delta \mathbf{u}) \rangle = \langle \delta \mathbf{u} \cdot ([\nabla \mathbf{u}]^T \cdot \boldsymbol{\nu} - \mathbf{u} \cdot \nabla \boldsymbol{\nu}) \rangle, \quad (29)$$

the sixth term is

$$\left\langle \boldsymbol{\nu} \cdot \left(-\frac{1}{Re} \nabla^2 \delta \mathbf{u} \right) \right\rangle = - \left\langle \frac{1}{Re} \delta \mathbf{u} \cdot \nabla^2 \boldsymbol{\nu} \right\rangle, \quad (30)$$

and finally the last term is

$$\begin{aligned} \langle \pi \nabla \cdot \delta \mathbf{u} \rangle &= \langle \nabla \cdot \pi \delta \mathbf{u} \rangle - \langle \delta \mathbf{u} \cdot \nabla \pi \rangle \\ &= - \langle \delta \mathbf{u} \cdot \nabla \pi \rangle, \end{aligned} \quad (31)$$

where π has to be periodic as $\langle \delta \mathbf{u} \cdot \hat{\mathbf{z}} \rangle \neq 0$ (a change in the mass flux is permitted for constant pressure-drop driven flow) for the surface term to be zero. Combining all these gives

$$\begin{aligned} \int_0^T \left\langle \frac{\delta \mathcal{L}}{\delta \mathbf{u}} \cdot \delta \mathbf{u} \right\rangle &= \langle \delta \mathbf{u}(\mathbf{x}, T) \cdot \{ \mathbf{u}(\mathbf{x}, T) + \boldsymbol{\nu}(\mathbf{x}, T) \} \rangle \\ &+ \langle \delta \mathbf{u}(\mathbf{x}, 0) \cdot \{ \lambda \mathbf{u}(\mathbf{x}, 0) - \boldsymbol{\nu}(\mathbf{x}, 0) \} \rangle \\ &+ \int_0^T \left\langle \delta \mathbf{u} \cdot \left\{ -\frac{\partial \boldsymbol{\nu}}{\partial t} - ([\mathbf{u}_{\text{lam}} + \mathbf{u}] \cdot \nabla) \boldsymbol{\nu} + \boldsymbol{\nu} \cdot (\nabla [\mathbf{u}_{\text{lam}} + \mathbf{u}])^T \right. \right. \\ &\quad \left. \left. - \nabla \pi - \frac{1}{Re} \nabla^2 \boldsymbol{\nu} \right\} \right\rangle dt. \end{aligned} \quad (32)$$

For this to vanish for all allowed $\delta \mathbf{u}(\mathbf{x}, T)$, $\delta \mathbf{u}(\mathbf{x}, 0)$ and $\delta \mathbf{u}(x, t)$ with $t \in (0, T)$ means

$$\frac{\delta \mathcal{L}}{\delta \mathbf{u}(\mathbf{x}, T)} = \mathbf{0} \quad \Rightarrow \quad \mathbf{u}(\mathbf{x}, T) + \boldsymbol{\nu}(\mathbf{x}, T) = \mathbf{0}, \quad (33)$$

$$\frac{\delta \mathcal{L}}{\delta \mathbf{u}(\mathbf{x}, 0)} = \mathbf{0} \quad \Rightarrow \quad \lambda \mathbf{u}(\mathbf{x}, 0) - \boldsymbol{\nu}(\mathbf{x}, 0) = \mathbf{0}, \quad (34)$$

$$\frac{\delta \mathcal{L}}{\delta \mathbf{u}} = \mathbf{0} \quad \Rightarrow \quad \frac{\partial \boldsymbol{\nu}}{\partial t} + (\mathbf{u}_{\text{lam}} + \mathbf{u}) \cdot \nabla \boldsymbol{\nu} - \boldsymbol{\nu} \cdot (\nabla [\mathbf{u}_{\text{lam}} + \mathbf{u}])^T + \nabla \pi + \frac{1}{Re} \nabla^2 \boldsymbol{\nu} = \mathbf{0}. \quad (35)$$

This last equation is the *dual* (or adjoint) Navier-Stokes equation for evolving $\boldsymbol{\nu}$ *backwards* in time because of the negative diffusion term. This dual equation has the same means of driving - constant pressure drop - as the physical problem, a situation also true for the constant mass-flux situation [104, 105].

The approach for tackling this optimization problem - ‘nonlinear adjoint looping’ - is iterative as in the linear situation [81, 2, 25, 82, 55] (see also the review [83]), the nonlinear calculation of [141, 142] using the (parabolic) boundary layer equations and more generally [56]. It is essentially as outlined in §2.

Step 0. Choose an initial condition $\mathbf{u}^{(0)}(\mathbf{x}, 0)$ such that

$$\left\langle \frac{1}{2} |\mathbf{u}^{(0)}(\mathbf{x}, 0)|^2 \right\rangle = E_0. \quad (36)$$

The (better) next iterate $\mathbf{u}^{(n+1)}(\mathbf{x}, 0)$ is then constructed from $\mathbf{u}^{(n)}(\mathbf{x}, 0)$ as follows:

Step 1. Time integrate the Navier-Stokes equation (21) forward with incompressibility $\nabla \cdot \mathbf{u} = 0$ and using the boundary conditions (16) from $t = 0$ to $t = T$ with the initial condition $\mathbf{u}^{(n)}(\mathbf{x}, 0)$ to find $\mathbf{u}^{(n)}(\mathbf{x}, T)$.

Step 2. Calculate $\boldsymbol{\nu}^{(n)}(\mathbf{x}, T)$ using (33) which is then used as the *initial* condition for the dual Navier-Stokes equation (35).

Step 3. Backwards time integrate the dual Navier-Stokes equation (35) with incompressibility (24) and boundary conditions (16) from $t = T$ to $t = 0$ with the ‘initial’ condition $\boldsymbol{\nu}^{(n)}(\mathbf{x}, T)$ to find $\boldsymbol{\nu}^{(n)}(\mathbf{x}, 0)$.

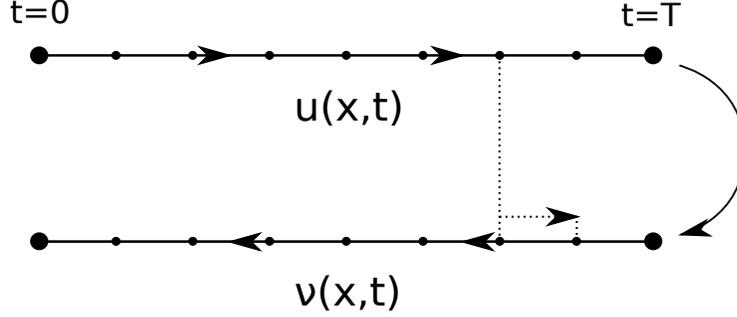


Figure 4. Checkpointing: during the (backward-in-time) calculation of $\nu(\mathbf{x}, t)$ (indicated by the lower line), the velocity $\mathbf{u}(\mathbf{x}, t)$ is recalculated in short sections from each checkpoint (shown as dots on the forward-in-time calculation) where $\mathbf{u}(\mathbf{x}, t)$ has been stored during the forward-in-time calculation.

Step 4. Use the fact that

$$\frac{\delta \mathcal{L}}{\delta \mathbf{u}(\mathbf{x}, 0)} = \lambda \mathbf{u}(\mathbf{x}, 0) - \nu(\mathbf{x}, 0) \quad (37)$$

is now computable to move $\mathbf{u}(\mathbf{x}, 0)$ towards a maximum of \mathcal{L} . One approach [104, 105, 108] is to simply move $\mathbf{u}^{(n)}(\mathbf{x}, 0)$ in the direction of maximum ascent of \mathcal{L} , i.e. a correction to $\mathbf{u}^{(n)}$ is calculated as follows:

$$\mathbf{u}^{(n+1)} = \mathbf{u}^{(n)} + \epsilon \left[\frac{\delta \mathcal{L}}{\delta \mathbf{u}(\mathbf{x}, 0)} \right]^{(n)} \quad (38)$$

$$= \mathbf{u}^{(n)} + \epsilon (\lambda \mathbf{u}^{(n)}(\mathbf{x}, 0) - \nu^{(n)}(\mathbf{x}, 0)), \quad (39)$$

with λ chosen such that

$$E_0 = \left\langle \frac{1}{2} |\mathbf{u}^{(n+1)}(\mathbf{x}, 0)|^2 \right\rangle \quad (40)$$

$$= \left\langle \frac{1}{2} |(1 + \epsilon \lambda) \mathbf{u}^{(n)}(\mathbf{x}, 0) - \epsilon \nu^{(n)}(\mathbf{x}, 0)|^2 \right\rangle. \quad (41)$$

Here ϵ is a parameter which can be adjusted as the iteration proceeds to improve convergence (e.g. [105, 108]). (The same procedure can be used to solve the linearised problem but with step 4 replaced by simply setting $\mathbf{u}(\mathbf{x}, 0)$ equal to $\nu(\mathbf{x}, 0)$ or a rescaled version of it: this is known as the power method - e.g. see [25].)

Other strategies have been adopted - e.g. a relaxation approach [86, 37] or a conjugate gradient method [18, 19, 20, 21, 69] - but it is presently unclear which, if any, is superior.

The iterations are repeated until some convergence criterion is fulfilled. Some authors [104, 105, 108] have concentrated on how the residual $\langle (\delta \mathcal{L} / \delta \mathbf{u}(\mathbf{x}, 0))^2 \rangle$ behaves as a function of the iteration number and others [18, 19, 20, 21] have focussed on the incremental change in \mathcal{L} between iterations. The former seems more natural given the latter depends on how large a step is taken in moving $\mathbf{u}(\mathbf{x}, 0)$ but has its issues too

(e.g. Figure 9 in [105], Figure 9(b) in [108]). More work is needed to identify a robust convergence criterion.

There is one further practical issue which needs to be discussed when the fully nonlinear optimisation problem is considered: the dual Navier-Stokes equation is linear in $\boldsymbol{\nu}$ but depends on $\mathbf{u}(\mathbf{x}, t)$. This field either needs to be stored in totality (over the whole volume and time period), which is only practical for low resolution, short integrations, or must be recalculated piecemeal during the backward integration stage. This latter ‘check-pointing’ approach [8, 59] requires that \mathbf{u} is stored at regular intermediate points, e.g. $t = T_i := iT_{opt}/n$ for $i = 1, \dots, n - 1$, during the forward integration stage. Then to integrate the adjoint equation backward over the time interval $[T_i, T_{i+1}]$, \mathbf{u} is regenerated starting from the stored value at $t = T_i$ by integrating the Navier-Stokes equations forward to T_{i+1} again: see Figure 4. The extent of the check pointing is chosen such that the storage requirement for each subinterval is manageable. The extra overhead of this technique is to redo the forward integration for every backward integration, so approximately a 50% increase in cpu time, assuming forward and backward integrations take essentially the same time. As memory restrictions may make full storage impossible, this is a small price to pay.

3.2. Results: Minimal seeds and routes to turbulence

In fluid mechanics, the kinetic energy of the perturbation \mathbf{u} (defined in (17)) has invariably been used as the norm to constrain the competitor initial conditions in the optimisation problem and typically also used as the objective functional to be optimised (see [48] for a counterexample). However a different objective functional can be chosen provided it clearly takes on larger values in the turbulent state than those reached in the basin of attraction of the laminar state (e.g. viscous dissipation rate or even the total viscous dissipation over the period $[0, T]$ - [86, 37]).

After these choices, the optimisation problem has two operational inputs - the time horizon T and the initial ‘distance’ d (read energy E_0 for the rest of this section) - beyond a complete specification of the system parameters (e.g. Re and length of pipe in pipe flow).

3.2.1. Large T The approach for identifying the minimal seed outlined above is to choose a large fixed value of T and to explore how the global optimal value behaves as E_0 is increased from zero [104, 105, 108] (complementary work [86, 37] instead reduces E_0 until no turbulent state is found). This was first pursued for the full Navier-Stokes equations in pipe flow [104] where the linear ($E_0 \rightarrow 0$) energy growth optimal perturbation - hereafter referred to as the ‘LOP’ - is 2-dimensional (streamwise-independent). The optimal for small but finite E_0 is a smooth nonlinear adjustment of the linear result retaining its 2 dimensionality and with a slightly reduced energy gain (hereafter referred to as a ‘QLOP’ - *quasi linear optimal perturbation*). Beyond an initial critical energy E_{3d} , however, a new completely different 3-dimensional

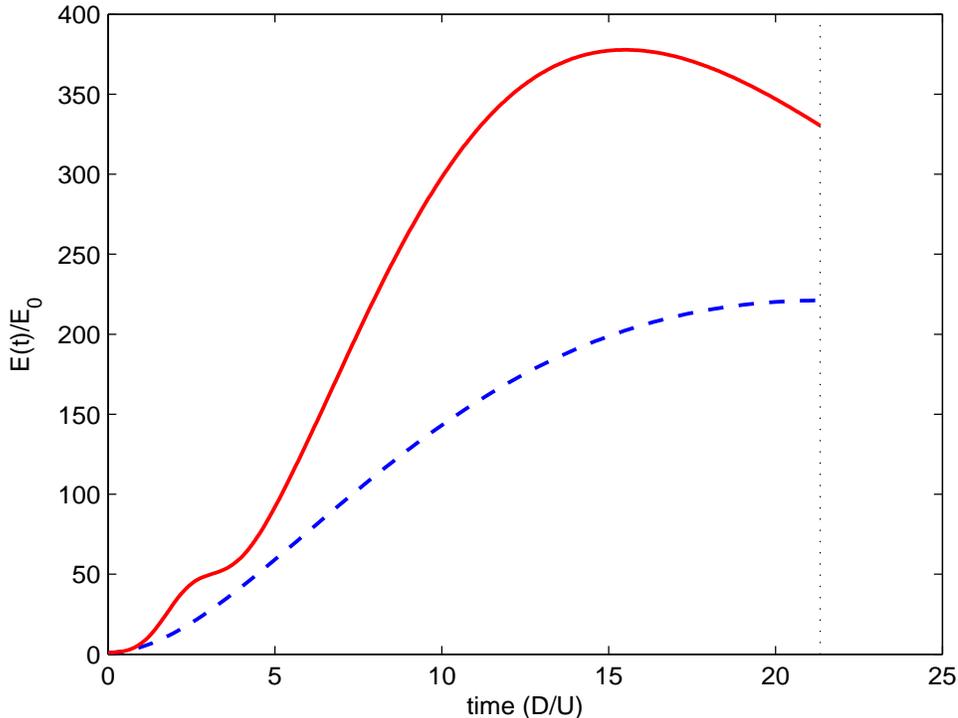


Figure 5. Pipe flow: the evolution of the linear and nonlinear energy growth optimals at $Re = 1750$ in a pipe of length π radii and constant flux [104]. The blue dashed line corresponds to the linear optimal for $E_0 \rightarrow 0$ whereas the red solid line is the nonlinear optimal for $\approx 1.5E_{3d}$ both calculated using (a short) T equal to the linear growth optimal time shown as a vertical black dotted line (the nonlinear optimal actually produces even more growth at a slightly earlier time).

perturbation was found as the global optimal and christened the ‘NLOP’ (*nonlinear optimal perturbation*) [104]: see Figure 5. This NLOP reflects a clear strategy by the fluid to spatially localise the starting perturbation so that its peak amplitude is larger but only over a limited volume to cheat the global energy constraint. The original pipe geometry used in [104] was very short so the NLOP is only localised in the radial and azimuthal directions (see Figure 2(a) in [104]). Subsequent computations in much longer pipe geometries have confirmed that the NLOP fully localises by also localising in the axial (streamwise) direction too [105, 106]: see Figure 7. Physically, the NLOP evolves by initially ‘unpacking’ (delocalising) under the influence of the background shear flow and taking advantage of 3 distinct well-known linear mechanisms for transient growth: the Orr mechanism which occurs quickly, oblique wave growth which operates over an intermediate timescale, and lift-up which occurs over a slow timescale (see [105, 37] and references therein). These mechanisms are unrelated in the linear problem i.e. initial conditions exploiting each are distinct from each other so that the mechanisms do not

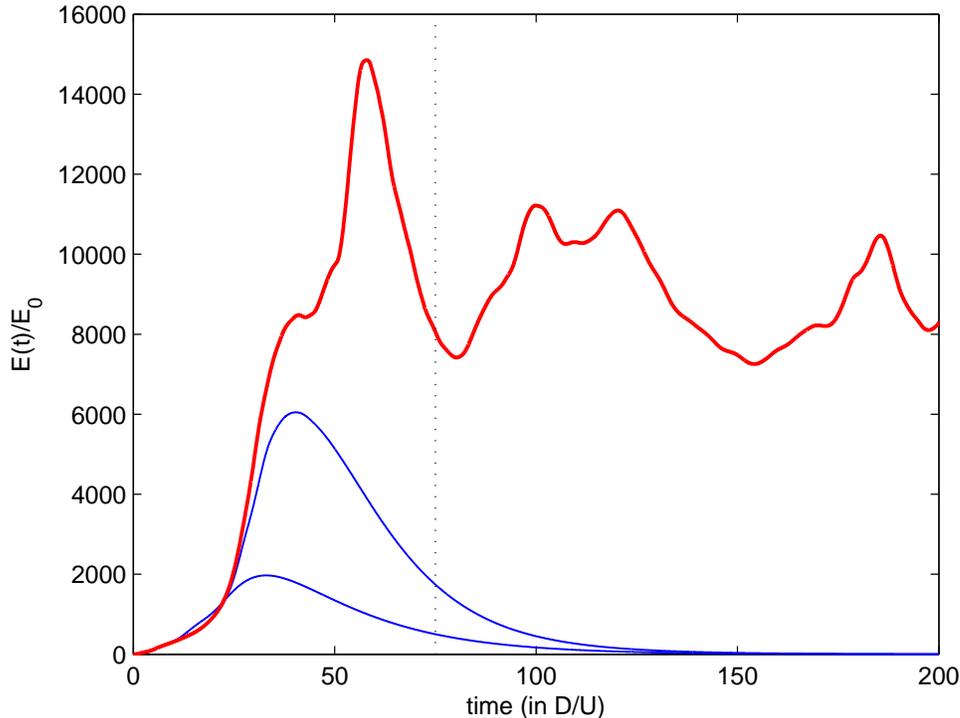


Figure 6. Pipe flow: the evolution of the energy growth optimals at $Re = 2400$ in a constant-flux pipe of length 10 radii using a ‘large’ time $T = 75 D/U$ indicated by a vertical black dotted line (see section 5 of [105]). The thin blue lines correspond to the NLOPs for two E_0 s just below E_c (within 1%) with turbulence not triggered. The thick red solid line is for E_0 just above E_c (again within 1%) and turbulence is now clearly triggered (see Figure 7 for snapshots of this evolution).

combine. However they can communicate with the addition of nonlinearity which allows each mechanism to ‘pump prime’ the next generating a much higher overall growth [105, 37]. Figure 5 shows a period of growth in the NLOP curve (the oblique wave growth stage) which terminates at $t \approx 2.5$ before further growth (lift up) occurs until $t \approx 16$ (the Orr mechanism occurs over a much faster timescale - $O(0.1)$ see Figure 1 of [105] - and is therefore hidden in this plot). Appendix B discusses a very simple model to illustrate this phenomenon (compare Figures 5 and 9).

As E_0 is increased further, this NLOP remains the global optimal until another critical energy E_{fail} is reached beyond which the optimisation procedure fails to converge. Pringle et al. [105] interpret (their conjecture 1) that this corresponds to the first energy at which an initial condition can reach the turbulent state since then the extreme sensitivity of the final-state energy at T to changes in the initial condition, due to exponential divergence of adjacent states will effectively mean non-smoothness and prevent convergence. That the turbulent state has been reached at E_{fail} is easily

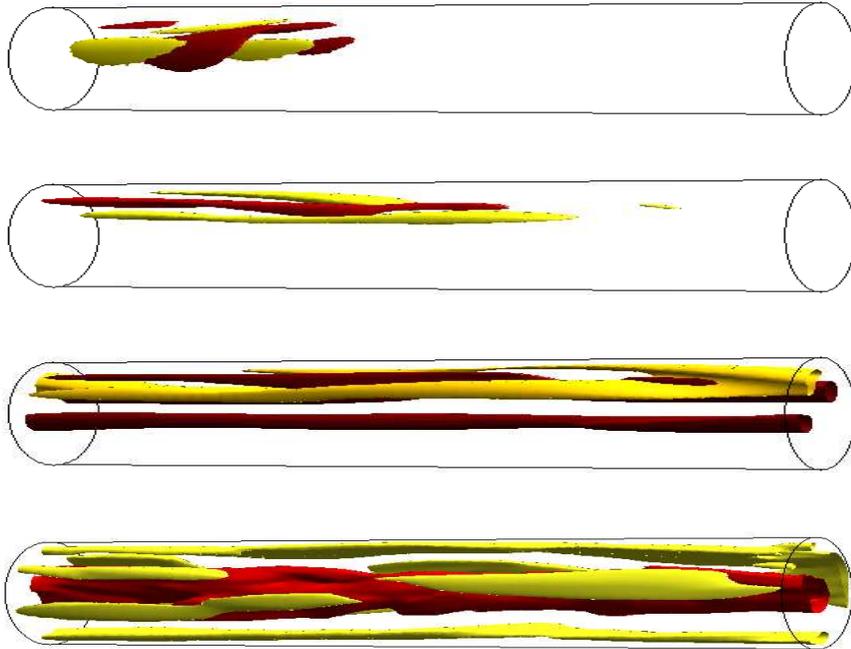


Figure 7. Pipe flow: snapshots showing the evolution (time running downwards and fluid flow left to right) of a turbulence-triggering perturbation which approximates the minimal seed in a 10 radii long pipe at $Re = 2400$ (constant flux) [105] (the corresponding evolution of the total energy is shown as the red thick line in Figure 6). The isocontours in each plot correspond to 50% of the maximum (light/yellow) and 50% of the minimum (dark/red) of the streamwise perturbation velocity in the pipe at $t = 0$ (top: essentially the minimal seed), $t = 5$ (second down), $t = 20$ (third down) and $t = 75 D/U$ (bottom: the target time). The perturbation energy is initially localised but quickly spreads out to generate streamwise streaks (by $t \approx 10$) which then break up to generate turbulence.

confirmed by examining the endstates $\mathbf{u}(\mathbf{x}, T)$ reached as part of the iteration algorithm so that E_{fail} is at least an upper bound on the true critical energy E_c , which is the lowest energy at which turbulence can be triggered. Arguing that in fact $E_{fail} = E_c$ requires a belief or hope that the search algorithm will find any state on the (initial) energy hypersurface which can reach the turbulent state if it exists. This is never likely to be proved but can at least be made plausible by rerunning the algorithm with a variety of initial starting states (e.g. see Figure 14 in [105]).

Pringle et al. [105] also noticed (their conjecture 2) that the NLOP converged to the minimal seed as $E_0 \rightarrow E_c^-$ for their choice of the energy functional. It now seems clear that there is nothing special about using the perturbation energy as an objective functional merely that it is one of a class of functionals which take on heightened values for turbulent flows. Providing such a functional is chosen, the best way to maximise it should be to get as close to the basin boundary as possible while remaining on the

E_0 hypersurface since all other states will be sucked into the laminar state if T is chosen larger than the typical transient growth time. This is certainly borne out by comparing estimates for E_c and the corresponding minimal seed found in plane Couette flow using the total viscous dissipation over $[0, T]$ [86] or the final energy growth [108] as objective functionals. One of the motivations for choosing the total dissipation over the time period in [86, 37] was to capture the fastest transition. Finding the critical energy, however, is a problem in finding just one state which triggers turbulence so, conceptually, the final viscous dissipation rate could have been used just as well (although practically, time averaging the dissipation rate helped smoothen the effect of dealing with turbulent endstates in their optimisation algorithm).

3.2.2. Optimising over T Given the need to choose T ‘large enough’, one natural idea has been to ‘optimise it out of the problem’ by asking the question ‘for a given initial perturbation of energy E_0 what is the largest growth over *any* T ?’ Algorithmically, this requires only a small change but can dramatically alter the results for $E_0 < E_c$. Work by [108] in plane Couette flow has shown how a QLOP achieving large growth at small times can ‘mask’ the emergence of a NLOP (which has a longer growth time) as E_0 increases (see their Figure 2). When E_c is reached, however, there is the same leap in optimal value which is now also accompanied with a sudden leap in the optimal time T (e.g. see Figures 3 and 10 in [108]). Operationally, this means that the minimal seed (and E_c) can still be found albeit only when the algorithm fails to converge rather than as a smooth convergence procedure as E_0 approaches E_c from below.

3.2.3. Small T Cherubini and coworkers [18, 19, 20] have computed the nonlinear optimal energy growth disturbances for small T in boundary layer flow. This is a particularly challenging system since the flow is spatially developing and open (flow leaving the computational domain is not recycled using periodic boundary conditions) making longer time runs very expensive. Rather than isolating minimal seeds[†], their focus has been to establish that nonlinear mechanisms can enhance transition by uncovering energies E_0 where the NLOP subsequently triggers turbulence (when integrated beyond T) whereas the linear optimal, scaled up in energy, does not. By analysing the transition path of their NLOP in detail, they find a number of mechanisms believed generic to boundary layer transition [19, 20]. Some of these appear to carry over to plane Couette flow [21] when transition is studied again using short-time optimisation analysis. Here, for small T , at least two scenarios are found: a ‘highly dissipative’ bursting path and short-path depending on how close the perturbation is initially to the basin boundary.

[†] Note Cherubini and coworkers [18, 19, 20] use the term ‘minimal seed’ differently to this article: they use it to describe the minimal spatial structure found in their NLOPs whereas here it is the minimal energy state, when infinitesimally disturbed, to trigger turbulence.

3.3. Summary

The key points to note in applying this variational approach in fluid mechanics are as follows.

- (i) The choice of objective functional is not important provided it takes on heightened values for turbulent flows compared to the laminar state. The algorithm works best when there is a good separation between functional values attained within the basin of attraction and the turbulent state. This is typically the case for final time values of the perturbation kinetic energy or viscous dissipation rate when the flow domain is large enough, and Re is not too close to the critical value at which the turbulent state first appears [105].
- (ii) When seeking the transition threshold E_c , T needs to be large enough to avoid two problems: a) possible transients with large growth caused by some parts of the basin boundary attaining large functional values, and b) to allow sufficient times for states to reach the turbulent attractor [105, 108].
- (iii) The minimal seed can either be identified smoothly as a converged NLOP for $E_0 \rightarrow E_c^-$ at fixed but large T [105] or as the first initial condition encountered by the algorithm which triggers turbulence as E_0 is increased if T is being optimised over [108].

Finally, we end this section with a note of caution. The approach relies on the correct identification of the global optimal to a (non-convex) nonlinear optimisation problem. There is unlikely to ever be a formal criterion to verify that this has been accomplished (in contrast to the linearised problem) so care must be taken to collect supporting evidence for this claim if possible. So far, the results obtained have been checked for consistency in a number of different ways:

- (i) Robustness: it is clearly good practice to check global optimality by seeding the variational algorithm with a suite of very different initial states to see if the same optimal emerges each time (e.g. Figures 14 and 15 of [105]) or, if not, that the new optimal is just a local optimal.
- (ii) Physical plausibility: it is reassuring that the NLOPs found are localised and appear to ‘join up’ the various hitherto-unconnected linear energy-growth mechanisms during their evolution [105, 19, 37].
- (iii) Using different objective functionals: the cross-checking of results between two groups [86, 108] using different algorithms and objective functionals has been crucial in establishing the feasibility of this approach to realistic problems in fluid mechanics.
- (iv) Of all the states on the basin boundary, the minimal seed should experience the largest increase in ‘distance’ (read perturbation energy) as it evolves up to the edge state and states ‘close’ to the minimal seed should evolve to pass close to the edge state before either relaminarising or trigger turbulence (e.g. Figure 17 of [105]).

4. Applications

4.1. Climate modelling and Weather Forecasting

In climate modelling and weather forecasting, the sensitivity of predictive models to uncertainty in their initial conditions (the current best guess of the model’s state) is of central importance. Lorenz [80] was the first to introduce the concept of singular vectors as a way to analyse how this uncertainty (presumed small) may grow with time. However, it wasn’t until the late 1980s when the idea really took hold following the work of Farrell [44, 45] in atmospheric and the accompanying realisation in general shear flows of the phenomenon of (linear) transient growth or (linear) ‘optimal perturbations’ [12, 43, 57, 15, 130, 111]. This quickly led to the implementation of singular vectors to generate ensemble forecasts (e.g. at the European Centre for Medium-Range forecasting (ECMWF) [100, 14]).

The initial uncertainties, however, needn’t be small and their true behaviour with time may differ significantly from that given by the evolution operator linearised around the base solution. As a result, there have been attempts to include some nonlinearity into the problem by an iterative approach [98, 6] as well as a proposal for a fully nonlinear approach by Mu and coworkers: [87, 88, 90, 91, 92] and more recently [35, 140, 66]. [88] in particular introduced the concept of a ‘conditional nonlinear optimal perturbation’ (CNOP), which is exactly the NLOP discussed above, to study the predictability of a very simple coupled atmosphere-ocean model. The approach has subsequently been applied to other weather and climate events (e.g. El Niño events [138, 95, 34], ‘blocking’ onset [65, 91, 94] and the Kuroshio large meander [134, 135]) and extended to treat uncertainties in a model’s parameters as well [93]. As mentioned in the Introduction, the idea to study the transitions between different stable states was also suggested by Mu and co-workers in the context of ocean modelling where multiple equilibria are a generic feature of general circulation models. To illustrate this idea, [89] studied a very simple box model of the thermohaline circulation consisting of 2 ODEs using CNOPs. Subsequent work has tried to scale up the calculation of CNOPs to realistic PDE models with [90] computing CNOPs for a 2D quasigeostrophic model with 512 grid points and [126] treating a 2D barotropic double-gyre ocean flow model with 4800 degrees of freedom. The latter study, however, concluded that finding basin boundaries was just too computationally expensive to attempt. Given the recent successes in the transition problem, this conclusion probably deserves to be revisited.

Before leaving this section, the work of Toth and Kalnay [128, 129, 71] on the so called ‘breeding’ method deserves mention. This consists of adding a small arbitrary perturbation to the full forecasting model, allowing this to evolve and then rescaling it after a given time to reseed the next forecast period. This procedure, which breeds ‘bred vectors’, is less optimal in identifying optimal perturbations for a given period (in fact perturbations which emerge have *just* grown the fastest rather than *will* grow the fastest) but is readily generalised to incorporate the finite-amplitude nature of the uncertainties since the full forecasting model is used. This method is used to generate ensemble

forecasts at the National Centers for Environmental Prediction (NCEP) (formerly the US National Meteorological Center).

4.2. *Thermoacoustics*

The idea of using an optimisation approach to locate a basin boundary was also introduced to the field of thermoacoustics at the same time as in the transition to turbulence community. [69] treated a simple model of thermoacoustic system - a horizontal Rijke tube - where the laminar state is a fixed point, the edge state is an unstable periodic orbit and the ‘turbulent’ state is a stable periodic orbit (respectively the ‘lower’ and ‘upper’ branches which emerge from a saddle node bifurcation). In contrast with the strategy outlined above, [69] found the minimal seed by looking for the minimal energy state to reach the (unique) edge state at intermediate times rather than the minimal energy state that reaches the stable periodic orbit at long times. However, the result is the same and was confirmed in [70] using the latter strategy. As in the transition problem (with now the nonlinear terms not energy-preserving), the minimal seed is found to be completely different from the linear optimal. The Rijke tube system is sufficiently simple (a couple of time-dependent 1-space dimension PDEs) to perform an exhaustive survey of transient growth possibilities over amplitude and event horizon T [70] as well as including noise [136]. It is also realistic enough to achieve some correspondence with experiments [64].

4.3. *Control*

Work in controlling fluid flows has long flirted with fully nonlinear methods (e.g. [67, 68, 56, 9, 23, 103]) but they remain currently very expensive and probably still impractical [75]. The recent success in identifying NLOPs and minimal seeds, however, has led to some new activity in this direction. [101] has recently used the adjoint-based optimisation procedure discussed here for the full Navier-Stokes equations to control the 2D boundary layer dynamics over a bump by blowing and sucking appropriately through the boundaries. An extension to 3D has been carried out in [22] where initial conditions corresponding to the LOP and NLOP have been treated. This work is still some way short of practical application because it relies on knowing the full state of the system at each time and costly optimisation computations which can’t be done in real time but it is at least starting to identify the most promising physical mechanisms for control.

In a slightly different vein, a more *nonlinearly stable* plane Couette flow has been designed by imposing spanwise oscillations on the usual streamwise boundary shearing [109]. This work builds upon the fact that if the critical energy of the minimal seed can be found then new boundary conditions can be designed to increase this energy thereby improving the stability of the base state. While the choice of imposing spanwise oscillations was motivated by a large body of experimental and theoretical work (see [109] for references), it also meant that the base state was no longer steady but time-periodic. This has implications for the optimisation procedure which now not only has

to search for the optimal initial perturbation but also the exact time (or phase) during the base flow period when it should be introduced.

4.4. *Magnetic field generation & Mixing*

[137] treats the kinematic dynamo problem looking for the velocity field of an electrically-conducting fluid which produces the greatest growth of magnetic field at the end of a time interval T . The set of competitor fields is constrained either by the total energy or the L_2 norm of their strain rate (or equivalently the viscous dissipation rate) only. By using the optimisation procedure described here, a lower bound on the magnetic Reynolds number is identified for a dynamo which is only a 1/5th of that possible within the well-studied ABC-class of flows [24, 1].

Recently [49] has considered optimal mixing in 2D channel (plane Poiseuille) flow using a nonlinear optimisation approach. The mixing of a passive scalar, initially arranged in two layers, is considered in a parameter regime where the flow is linearly stable. Nonlinear-adjoint looping is used to identify optimal perturbations which lead to maximal mixing and the classical Taylor dispersion mechanism (where shear enhances dispersion) is found to emerge naturally from the calculations.

Finally, we close this section by noting that the optimisation procedure discussed in this article is closely related to the well known data assimilation procedure wherein a dynamical model of, and incomplete observations about, a real time-evolving system are used to constrain the initial state of the model such that the ‘best’ solution over a given time horizon can be sought. This is achieved by minimising an appropriate cost functional which penalises the deviations of the predicted solution away from known observations and possible uncertainties in the dynamical model. In contrast, the optimisation approach discussed here assumes a perfect dynamical model and seeks to maximise an objective functional over all initial conditions of a certain size in the absence of constraining observations. Data assimilation is used extensively in many areas of the geosciences (such as weather forecasting e.g [31, 71], oceanography [7] and more recently modelling the Earth’s dynamo [50]).

5. Final summary and future directions

This article has been a pedagogical introduction to an optimisation technique which offers a new way to probe the basin boundary of a state in a dynamical system. Although the discussion has concentrated on this well-defined situation for clarity of exposition, the technique can also usefully be employed for systems with just one global attractor and at least one long-lived but ultimately-repelling state (sheared fluid flows with enforced short wavelength dynamics are prime examples of this situation since then the turbulence seems only transient e.g. [42, 120]). Then the global attractor does not have a basin boundary but there is instead an ‘edge’ or manifold in phase space which

divides initial conditions which immediately converge to the global attractor and those that first visit one of the repellers. The same game can then be played providing a functional can be identified which is clearly maximised in the target repeller for a time long enough for the repeller to be reached yet shorter than the mean lifetime of that repeller. This is in fact probably the situation in all the fluid flow calculations done so far (e.g. the turbulence in the 5 diameter long pipe in [105] is actually only transient at $Re = 2400$ but the mean lifetime is so large $\gg 100D/U$ that it mimicks an attractor on the timescales of the calculations).

The optimisation technique involves maximising a functional which takes on much larger values in the target state than in the basin of attraction of the starting state subject to an initial amplitude constraint and other constraints which include that the governing equations are satisfied - see the summary in §3.3. The iterative approach to solving this variational problem is not new (e.g [56]) nor is the realisation that there should be a sudden jump in the optimal value as the initial amplitude increases to penetrate the basin boundary particularly profound. What is noteworthy, however, is that the technique appears to work for large degree-of-freedom discretizations of PDE systems approaching practical application. Typically in such systems, states have basin boundaries which may be very convoluted in places yet the success reported here indicates that this needn't be a serious problem. Whether this is because the minimal seed for the cases studied so far is away from such areas or that the variational procedure smoothens over these regions to give an approximate but still good answer remains to be clarified.

For the particular application studied here - transition to turbulence in shear flows - the optimisation approach provides a pleasing theoretical bridge between the two different theoretical perspectives of (linear) optimal perturbation theory and the (nonlinear) dynamical system approach. It is now much clearer how nonlinearity interacts with the linear transient growth mechanisms to achieve transition at least close to the amplitude threshold. Furthermore, the first steps have been taken to actively use the technique to *design* more stable states by adjusting their driving slightly [109].

There are practical issues, however, needing development. The optimisation techniques being used to maximise the objective function given gradient information with respect to the initial conditions are typically simple-minded with no attempt made so far to tailor them to the system being treated. There is also no consensus as yet on what convergence criteria should be used or *a posteriori* checks to confirm that the global maximum has been found. The last issue is particularly important of course and an agreed level of care is needed.

5.1. Future directions

The optimisation approach is incredibly flexible and there is no reason why other information cannot be sought from a dynamical system. For example, [105] talks about identifying the peak instantaneous pressure in a transitional fluid flow, which is of key

concern in certain applications (e.g. pipeline structural integrity). Some preliminary calculations have also been done in the pre-turbulence regime to probe the existence of other solutions in phase space close to the laminar state (see Chapter 6 in [107]). The idea here is that optimising the energy after a long time should select initial conditions on the stable manifolds of nearby solutions *if* these intersect the initial-energy hypersurface (for ‘infinite’ time, this is clearly the only way to get a non-zero maximum). Results so far are encouraging: see Figure 6.14 of [107] which suggests that the spanwise-localised solution of [121] has been rediscovered using this approach. Ultimately, the hope is that this technique could be used to estimate when alternative solutions to the basic state emerge in phase space as Re is increased from 0.

So far only the basin boundaries of steady and time-periodic states have been investigated using this technique and with only one other attractor present. Treating multiple alternate attractors is a straightforward extension which just relies on defining the objective functional appropriately. Handling aperiodic states, however, is more troublesome. In the periodic state situation, both the structure of the minimal seed *and* exactly when it is applied during the periodic cycle of the state are important [109]. This precise specification cannot be extended to an aperiodic (e.g. chaotic) state since it is plainly impractical to optimise over all time. However, typically, the aperiodic state of interest is bounded and almost recurs which suggests a looser perhaps statistical formulation of the problem (e.g. the optimisation is performed over a representative sampling of the state). A more natural path in this situation and, in fact more generally, however, is to consider multiple disturbances. In the discussion so far, the assumption has been that the system is disturbed *once* and then evolves perfectly. In practice, disturbances to real systems are not isolated or indeed equally likely. Developing the optimisation technique to encompass these realities will obviously be important. Some tentative steps have already been made by considering multiple discrete disturbances [79] and there is an interesting connection to be made with ‘instanton’ theory for the continuous noise case (e.g.[46]). Some calculations done already have shown that adding noise to a system can still pick out the minimal seed route to ‘transition’ as outlined here [136].

Another way to making greater connection with experiments is to restrict the competitor set of initial states which acknowledges the fact that only a reduced subset of all initial conditions are achievable in the laboratory. This can be accomplished simply by projecting the gradient vector of the objective functional onto a reduced set of realisable initial conditions and looking for the minimal seed in this subclass of disturbances. Just such an approach, for example, could lower yet further the latest amplitude threshold scaling for pipe flow [102].

Finally, the initial work [109] seeking to improve the stability of the base state by manipulating the system is ripe for further development either from the perspective of keeping a flow laminar or, say, reducing the power consumption of an already turbulent flow. In the latter case, relaminarisation may not be achievable but one can still ask the question what boundary manipulation is best in reducing the turbulent energy

dissipation across a given (long) time interval. Now, of course, finding the basin boundary of the turbulent state is out of reach but the same variational machinery can be used to tackle the nonlinear minimisation problem.†

Hopefully, the power of the optimisation technique discussed here should be clear. Ever increasing computer power is making direct simulations of systems more common with the concomitant need to process and interpret this data crucial. The optimisation technique discussed here has a huge potential to help with this and should surely become a standard theoretical tool in the near future.

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Appendix A: The 2D linear transient growth problem

The dynamics in the simple ODE model (10) around the equilibrium $\mathbf{X}_0 = (0, 0)$ are linearised when $d \ll 1$. In this limit, the equation for \mathbf{x} is just

$$\frac{d\mathbf{x}}{dt} = L\mathbf{x}, \quad \text{where} \quad L := \begin{bmatrix} -1 & 10 \\ 0 & -10 \end{bmatrix} \quad (42)$$

and standard matrix manipulations can then calculate the maximal achievable distance after a time T . L has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -10$ and corresponding eigenvectors \mathbf{q}_1 and \mathbf{q}_2 . Defining the matrix $\mathbf{Q} := (\mathbf{q}_1 | \mathbf{q}_2)$ (the 2 eigenvectors arranged in columns) and

$$e^{\Lambda T} = \begin{bmatrix} e^{\lambda_1 T} & 0 \\ 0 & e^{\lambda_2 T} \end{bmatrix} \quad (43)$$

then, if \mathbf{a} is a vector specifying the initial condition

$$\mathbf{x}(0) = \sum_{j=1}^2 a_j \mathbf{q}_j = \mathbf{Q}\mathbf{a} \quad \Rightarrow \quad \mathbf{x}(t) = \sum_{j=1}^2 a_j e^{\lambda_j T} \mathbf{q}_j = \mathbf{Q}e^{\Lambda T} \mathbf{a}. \quad (44)$$

† The boundary manipulation is applied across the time interval to eliminate the time origin issue.

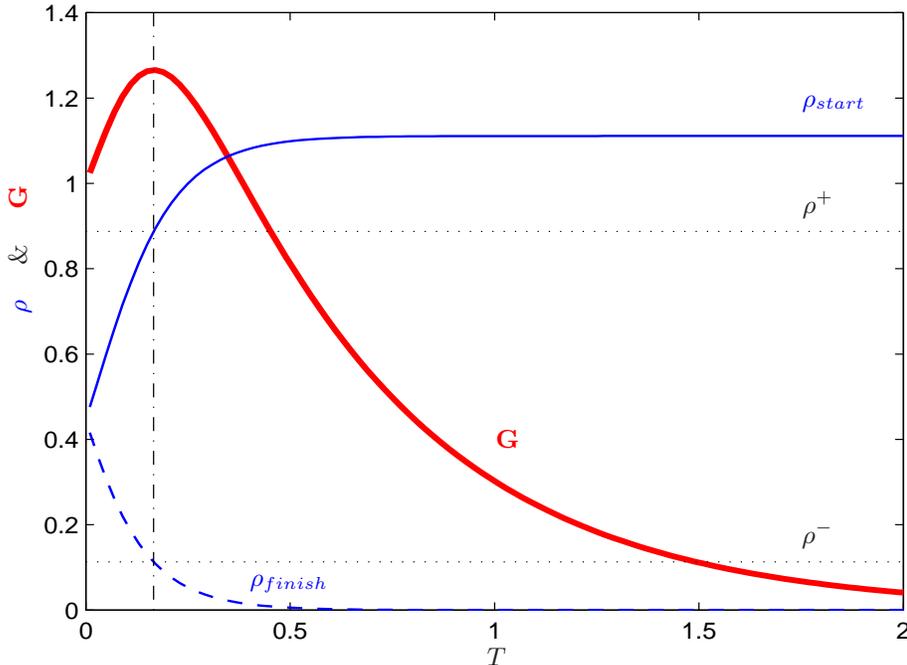


Figure 8. The maximum gain G (thick red line) against target time T for the linear system (42). The thinner solid/dashed blue lines indicate the starting/finishing values of $\rho := x_2/x_1$ for the corresponding optimal. Also shown as horizontal (black) dotted lines are the orientations where energy growth starts (ρ^+) and finishes (ρ^-) (ρ decreases in time for $x_2(0) > 0$). Notice that the maximum growth occurs precisely when the initial condition has $\rho(0) = \rho^+$ and T is such that $\rho(T) = \rho^-$.

Then $\max \mathcal{L} := G d^2$ where the gain

$$G(T) = \max_{\mathbf{a}} \frac{\|\mathbf{x}(T)\|_2^2}{\|\mathbf{x}(0)\|_2^2} = \max_{\mathbf{a}} \frac{\mathbf{a}^\dagger e^{\Lambda T} \mathbf{Q}^\dagger \mathbf{Q} e^{\Lambda T} \mathbf{a}}{\mathbf{a}^\dagger \mathbf{Q}^\dagger \mathbf{Q} \mathbf{a}} = \|\mathbf{M}\|_2^2 \quad (45)$$

and $\mathbf{M} := \mathbf{Q} e^{\Lambda T} \mathbf{Q}^{-1}$ (\dagger indicating transpose). G is therefore the largest singular value of \mathbf{M} or equivalently the largest (real) eigenvalue of the symmetric matrix $\mathbf{M}^\dagger \mathbf{M}$. The linear optimal $\theta = \theta^*$ increases monotonically from 0.1413 (25.4°) at $T = 0$, through 0.2322 (41.8°) at $T=0.16615$ where G peaks at 1.26590 , and then onto 0.2667 (48.0°) where $G \rightarrow 0$ as $T \rightarrow \infty$: see Figure 8.

In 2D, the situation is sufficiently simple to analyse completely for the general linear problem

$$\frac{d\mathbf{x}}{dt} = L\mathbf{x} := \begin{bmatrix} -a & b \\ 0 & -c \end{bmatrix} \mathbf{x} \quad (46)$$

with a, b and c all positive real numbers (the interesting stable case). Defining $\rho(t)$ as the orientation $x_2(t)/x_1(t)$ and energy $E(t) := \frac{1}{2}\mathbf{x}(t)^2$, energy growth begins at the

orientation ρ^+ and ends at the orientation ρ^- where

$$\rho^\pm := \frac{b \pm \sqrt{b^2 - 4ac}}{2c}. \quad (47)$$

The eigenvalues of L are $-a$ and $-c$ and with their corresponding eigenvectors, the general solution to (46) can be written down as

$$\mathbf{x}(t) = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-at} + \beta \begin{bmatrix} b/(a-c) \\ 1 \end{bmatrix} e^{-ct}. \quad (48)$$

Imposing the conditions that $\rho(0) = \rho^+$ and $\rho(T) = \rho^-$ requires

$$\frac{\beta}{\alpha} = \frac{\rho^+(c-a)}{c-a+b\rho^+}, \quad T^* := \frac{1}{c-a} \log \left[\left(\frac{\rho^+}{\rho^-} \right) \frac{c-a+b\rho^-}{c-a+b\rho^+} \right] \quad (49)$$

so that

$$G_{\max} = \frac{\rho^{+2}(\rho^{-2} + 1)}{\rho^{-2}(\rho^{+2} + 1)} e^{-2cT^*}. \quad (50)$$

Appendix B: Linear versus nonlinear transient growth

Here we discuss how nonlinear optimal perturbations are related to linear optimal perturbations when the nonlinearity in the system preserves the functional being optimised (a prime example being the energy-preserving nonlinearity of the Navier-Stokes equations in fluid mechanics). Consider an ODE system whose phase space is partitioned into subspaces each of which is invariant under the linearised dynamics about the origin $\mathbf{x} = \mathbf{0}$ which is a stable fixed point. If some of the subspaces support transient growth then it is possible for nonlinearities in the system to couple these growth processes to give much larger overall growth than any possible in the linearised dynamics. To illustrate this, we take a simple 4D system which has two such subspaces,

$$\dot{\mathbf{x}} := \frac{d\mathbf{x}}{dt} = \begin{bmatrix} -a & b & 0 & 0 \\ 0 & -c & 0 & 0 \\ 0 & 0 & -\epsilon a & \epsilon b \\ 0 & 0 & 0 & -\epsilon c \end{bmatrix} \mathbf{x} + \begin{bmatrix} -x_1 x_4 \\ 0 \\ 0 \\ x_1^2 \end{bmatrix}. \quad (51)$$

For simplicity, the linear dynamics in each subspace are identical up to a change in timescale which means that the same maximum transient growth (G_{\max} given by (50)) occurs in both subspaces but at two different times (given by (49)); T^* for subspace $U_1 := \{(x_1, x_2, 0, 0) \mid x_1, x_2 \in \mathbb{R}\}$ and T^*/ϵ for subspace $U_2 := \{(0, 0, x_3, x_4) \mid x_3, x_4 \in \mathbb{R}\}$: see Figure 9. Minimal nonlinear terms are included designed to a) conserve energy (as in the Navier-Stokes equations) and b) to allow the faster energy growth in subspace U_1 to pump-prime the slower growth in subspace U_2 . This is clearly seen to occur for δ large enough when taking the optimal initial condition for U_1

$$\mathbf{x}(0) = (x_1, x_2, x_3, x_4) = \delta(1, \rho^+, 0, 0), \quad (52)$$

see Figure 9 for an example using $(a, b, c, \epsilon) = (1, 10, 2, 0.1)$ (for the linear problem in U_1 , $G_{\max} = 6.57$ and $T^* = 0.66$).

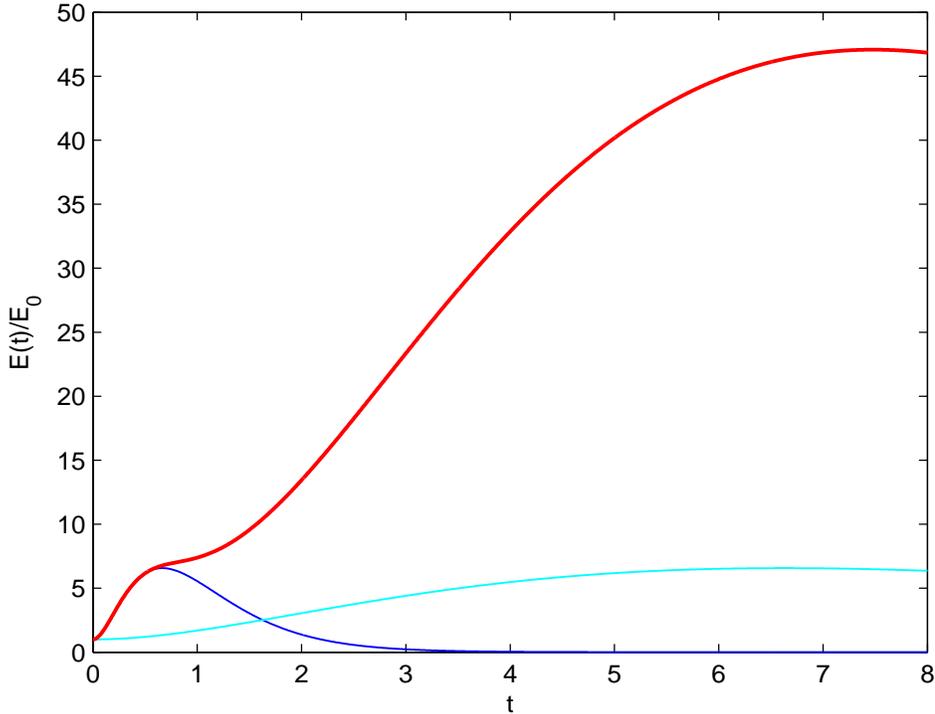


Figure 9. Linear and nonlinear transient growth for the system (51) with $(a, b, c, \epsilon) = (1, 10, 2, 0.1)$. The thin blue and cyan lines show optimal growth in U_1 and U_2 which both have a peak gain G of 6.57 but at different times $T^* = 0.66$ and T^*/ϵ respectively for infinitesimally small initial energies. The thick red line is the result of using the optimal initial condition (52) for the U_1 with $\delta = 0.15$ which is sufficient to allow the two transient growth processes to combine to produce much larger overall growth. Note the similarity of this nonlinear curve to that in Figure 5.

If the terms ‘LOP’ and ‘NLOP’ indicate the global linear and nonlinear optimals over asymptotically large T , both clearly must have

$$\dot{E}(0) = 0 \tag{53}$$

($E := \frac{1}{2}\mathbf{x}^2$) otherwise the time origin could be adjusted to increase the overall growth [26]. Since \dot{E} is only determined by the linear terms (the nonlinear terms can’t contribute as they are energy-preserving), then all the candidate initial conditions satisfying the constraint $\dot{E} = 0$ for the full nonlinear optimisation problem are actually present in the linear problem. However, the linear problem may never select the NLOP for *any* T because without nonlinearity its growth is overshadowed by other candidates (i.e. the NLOP may only be a local rather than global maximum or not even a maximum at all!). To illustrate this with a concrete example, consider a 6D extension of the above 4D system by adding another 2D subspace U_3 to $U_1 \oplus U_2$. If the linear transient growth in U_3 always produces slightly more growth over any choice of T than U_1 (at least until

the slower growth in U_2 takes over), the linear optimisation problem will never select any candidate initial condition which has some projection in U_1 . In contrast the NLOP will be the linear optimal for U_1 (i.e. (52)) since this is the only way $\dot{E}(0)$ vanishes non-trivially in U_1 . This situation is borne out by the NLOP found in (very high dimensional) pipe flow [104, 105]. In just 2D, however, there are only two candidates in the linear problem which are joint global optimisers. Hence in this case the NLOP has to be contained in the linear problem trivially.

References

- [1] Alexakis A 2011 Searching for the fastest dynamo: laminar ABC flows *Phys. Rev. E* **84** 026321.
- [2] Andersson P, Berggren M and Henningson D S 1999 Optimal disturbances and bypass transition in boundary layers *Phys. Fluids* **11** 134-150.
- [3] Babloyantz A and Destexhe A 1986 A low-dimensional chaos in an instance of epilepsy *Proc. Natl. Acad. Sci.* **83** 3513-3517
- [4] Baggett J S and Trefethen L N 1995 Low-dimensional models of subcritical transition to turbulence *Phys. Fluids* **9** 1043-1053.
- [5] Baggett J S, Driscoll T A and Trefethen L N 1997 A mostly linear-model of transition to turbulence *Phys. Fluids* **7** 833-838.
- [6] Barkmeijer J 1996 Constructing fast-growing perturbations for the nonlinear regime *J. Atmos. Sci.* **53** 2838-2851.
- [7] Bennett A F 1992 *Inverse Methods in Physical Oceanography* Cambridge University Press.
- [8] Berggren M 1998 Numerical solution of a flow-control problem: vorticity reduction by dynamic boundary action *Siam J. Sci. Comput.* **19** 829-860.
- [9] Bewley T R, Moin P and Temam R 2001 DNS-based predictive control of turbulence: an optimal benchmark for feedback algorithms *J. Fluid Mech.* **447** 179-225.
- [10] Biau D and Bottaro A 2004 Transient growth and minimal defects: two possible initial paths of transition to turbulence in plane shear flows *Phys. Fluids* **16** 3515-3529.
- [11] Biau D and Bottaro A 2009 An optimal path to transition in a duct *Phil. Trans. Roy. Soc. Lond. A* **367** 529-544.
- [12] Boberg L and Brosa U 1988 Onset of turbulence in a pipe *Z. Naturforsch.* **43a** 697-726.
- [13] Bottaro A, Corbett P and Luchini P 2003 The effect of base flow variation on flow instability *J. Fluid Mech.* **596** 113-142.
- [14] Buizza R and Palmer T N 1995 The singular-vector structure of the atmospheric global circulation *J. Atmos. Sci.* **52** 1434-1456.
- [15] Butler K M and Farrell B F 1992 3-Dimensional optimal perturbations in viscous shear-flow *Phys. Fluids* **4** 1637-1650.
- [16] Cantwell C D, Barkley D and Blackburn H M 2010 Transient growth analysis of flow through a sudden expansion in a circular pipe *Phys. Fluids* **22** 034101.
- [17] Chandrasekhar S 1961 *Hydrodynamic and Hydromagnetic Instability*, Oxford University Press.
- [18] Cherubini S, De Palma P, Robinet J-C and Bottaro A 2010 Rapid path to transition via nonlinear localised optimal perturbations in a boundary layer flow *Phys. Rev. E* **82** 066302.
- [19] Cherubini S, De Palma P, Robinet J-C and Bottaro A 2011 The minimal seed of turbulent transition in the boundary layer *J. Fluid Mech.* **689** 221-253.
- [20] Cherubini S, De Palma P, Robinet J-C and Bottaro A 2012 A purely nonlinear route to transition approaching the edge of chaos in a boundary layer *Fluid Dynam. Res.* **44** 031404.
- [21] Cherubini S and De Palma P 2013 Nonlinear optimal perturbations in a Couette flow: bursting and transition *J. Fluid Mech.* **716** 251-279.

- [22] Cherubini S, Robinet J-C and De Palma P 2013 Nonlinear control of unsteady finite-amplitude perturbations in the Blasius boundary-layer flow *J. Fluid Mech.* **737** 440-465
- [23] Chevalier M, Högberg M, Berggren M and Henningson D S 2002 Linear and nonlinear optimal control in spatial boundary layers *AIAA paper* **2002-2755**.
- [24] Childress S 1970 New solutions of the kinematic dynamo problem *J. Math Phys.* **11** 3063-3076
- [25] Corbett P and Bottaro A 2000 Optimal perturbations for boundary layers subject to streamwise pressure gradient *Phys. Fluids* **12** 120-130.
- [26] Cossu C 2005 An optimality condition on the minimum energy threshold in subcritical instabilities *C. R. Mecanique* **333** 331-336.
- [27] Cossu C, Chevalier, M and Henningson, D S 2007 Optimal secondary energy growth in a plane channel flow *Phys. Fluids* **19** 058107.
- [28] Cossu C, Brandt L, Bagheri, S and Henningson, D S 2011 Secondary threshold amplitudes for sinuous streak breakdown *Phys. Fluids* **23** 074103.
- [29] Cvitanović P and Gibson J F 2010 Geometry of turbulence in wall-bounded shear flows: periodic orbits *Phys. Scr.* **142** 014007.
- [30] Dauchot O and Manneville P 1997 Local versus global concepts in hydrodynamic stability theory *J. Phys. II France* **7** 371-389.
- [31] Daley R 1991 *Atmospheric data analysis* Cambridge University Press.
- [32] Drazin P G 2002 *Introduction to Hydrodynamic Stability* Cambridge University Press.
- [33] Drazin P G and Reid W H 1981 *Hydrodynamic Stability* Cambridge University Press.
- [34] Duan W, Yu Y, Xu H and Zhao P 2013 Behaviors of nonlinearities modulating the El Niño events induced by optimal precursory disturbances *Clim. Dyn.* **40** 1399-1413.
- [35] Duan W and Zhou F 2013 Non-linear forcing singular vector of a two-dimensional quasi-geostrophic model *Tellus* **65** 18452.
- [36] Duguet Y, Brandt L and Larsson B R J 2010 Towards minimal perturbations in transitional plane Couette flow *Phys. Rev. E* **82** 026316.
- [37] Duguet Y, Monokrousos A, Brandt L and Henningson D S 2013 Minimal transition thresholds in plane Couette flow *Phys. Fluids* **25** 084103.
- [38] Eckhardt B, Faisst H, Schmeigel A and Schumacher J 2002 Turbulence transition in shear flows. *Advances in Turbulence IX: Proceedings 9th European Turbulence Conference* (Southampton) (ed. I. P. Castro et al.), p701, CISME.
- [39] Eckhardt B, Schneider T M, Hof B and Westerweel J 2007 Turbulence transition in pipe flow *Ann. Rev. Fluid Mech.* **39** 447-468.
- [40] Ellingsen T and Palm E 1975 Stability of linear flow *Phys. Fluids* **18** 487-.
- [41] Ergäber H, Lenstra D, Krauskopf B, Wille E, Peil M, Fischer I and Elsasser W 2005 Mutually delay-coupled semiconductor lasers: Mode bifurcation scenarios *Opt. Commun.* **255** 286-296.
- [42] Faisst H and Eckhardt B 2004 Sensitive dependence on initial conditions in transition to turbulence in pipe flow *J. Fluid Mech.* **504** 343-352.
- [43] Farrell B F 1988 Optimal excitation of perturbations in viscous shear flow *Phys. Fluids* **31** 2093-2101.
- [44] Farrell B F 1989 Optimal excitation of baroclinic waves *J. Atmos. Sci.* **46** 1193-1206.
- [45] Farrell B F 1990 Small error dynamics and the predicability of atmospheric flows *J. Atmos. Sci.* **47** 2409-2416.
- [46] Freidlin M I and Wentzell A D 1998 *Random Perturbations of Dynamical Systems* Springer-Verlag, New York.
- [47] Foures D P G, Caulfield C P and Schmid P J 2012 Variational framework for flow optimisation using seminorm constraints *Phys. Rev E* **86** 026306.
- [48] Foures D P G, Caulfield C P and Schmid P J 2013 Localisation of flow structures using ∞ -norm optimisation *J. Fluid Mech.* **729** 672-701.
- [49] Foures D P G, Caulfield C P and Schmid P J 2014 Optimal mixing in two-dimensional plane Poiseuille flow at high Péclet number *J. Fluid Mech.* **748**, 241-277.

- [50] Fournier A, Hulot G, Jault D, Kuang W, Tangborn A, Gillet N, Canet E, Aubert J and Lhuillier F 2010 An introduction to data assimilation and predictability in geomagnetism *Space Sci. Rev.* **155** 247-291.
- [51] Gad-el-Hak M 2000 *Flow Control: Passive, Active and Reactive Flow Management*, Cambridge.
- [52] Gavarini M I, Bottaro A and Nieuwstadt F T M 2004 The initial stage of transition in pipe flow: role of optimal base-flow distortions *J. Fluid Mech.* **517** 131-165.
- [53] Gibson J F, Halcrow J and Cvitanović P 2008 Visualizing the geometry of state space in plane Couette flow *J. Fluid Mech.* **611**, 107-130.
- [54] Grossmann S 2000 The onset of shear flow turbulence *Rev. Mod. Phys.* **72** 603-618.
- [55] Guégan A, Schmid P J and Huerre P 2006 Optimal energy growth and optimal control in swept Hiemenz flow *J. Fluid Mech.* **566** 11-45.
- [56] Gunzburger M 2000 Adjoint equation-based methods for control problems in incompressible, viscous flows *Flow, Turbulence and Combustion* **65** 249-272.
- [57] Gustavsson L H 1991 Energy growth of 3-dimensional disturbances in plane Poiseuille flow *J. Fluid Mech.* **224** 241-260.
- [58] Heisenberg W 1924 Über stabilität und turbulenz von flüssigkeitsströmen *Ann. Phys. Lpz.* **74** 577-627.
- [59] Hinze M, Walther A and Sternberg J 2006 An optimal memory-reduced procedure for calculating adjoints of the instationary Navier-Stokes equations *Optim. Control Appl. Methods* **27** 19-40.
- [60] Hoepffner J, Brandt L and Henningson D S 2005 Transient growth on boundary layer streaks *J. Fluid Mech.* **537** 91-100.
- [61] Hopf E 1948 A mathematical example displaying features of turbulence *Commun. Appl. Maths* **1**, 303-322.
- [62] Hultgren L S and Gustavsson L H 1981 Algebraic growth of disturbances in a laminar boundary-layer *Phys. Fluids* **24** 1000-1004.
- [63] Itano T and Toh S 2001 The dynamics of bursting process in wall turbulence *J. Phys. Soc. Japan* **70** 703-716.
- [64] Jegadeesan V and Sujith R I 2013 Experimental investigation of noise induced triggering in thermoacoustic systems *Proceedings of the Combustion Institute* **34** 3175-3183.
- [65] Jiang Z and Wang D 2010 A study on precursors to blocking anomalies in climatological flows by using conditional nonlinear optimal perturbations *Quart. J. Roy. Meteor. Soc.* **136** 1170-1180.
- [66] Jiang Z N, Mu M and Luo D H 2013 A study of the North Atlantic Oscillation using conditional nonlinear optimal perturbation *J. Atmos. Sci.* **70** 855-875.
- [67] Joslin R, Gunzburger M D , Nicolaides R A, Erlebacher G and Hussaini M Y 1995 Self-contained automated methodology for optimal flow control *Institute for Computer Applications in Science and Engineering (ICASE)* technical report **95-64**.
- [68] Joslin R D, Gunzburger M D , Nicolaides R A, Erlebacher G and Hussaini M Y 1997 Self-contained automated methodology for optimal flow control validated for transition delay *AIAA Journal* **35** 816-824.
- [69] Juniper M P 2011 Triggering in the horizontal Rijke tube: non-normality, transient growth and bypass transition *J. Fluid Mech.* **667** 272-308
- [70] Juniper M P 2011 Transient growth and triggering in the horizontal Rijke tube *Int. J. Spray and Combustion Dynamics* **3** 209-224.
- [71] Kalnay E 2003 *Atmospheric Modelling, Data Assimilation and Predictability* Cambridge University Press.
- [72] Kawahara G, Uhlmann M and van Veen L 2012 The significance of simple invariant solutions in turbulent flows *Ann. Rev. Fluid Mech.* **44** 203-225.
- [73] Kerswell R R 2005 Recent progress in understanding the transition to turbulence in a pipe *Nonlinearity* **18** R17-R44.
- [74] Kerswell R R 2011 Lecture 9: Triggering Transitions - Towards Minimal Seeds, *Summer program in Geophysical Fluid Dynamics*, Woods Hole Oceanographic Institute, Massachusetts, USA

- (<http://www.whoi.edu/main//gfd/proceedings-volumes/2011>)
- [75] Kim J and Bewley T R 2007 A linear systems approach to flow control *Ann. Rev. Fluid Mech.* **39** 383-417.
 - [76] Kitson S and Geisow A 2002 Controllable alignment of nematic liquid crystals around microscopic posts: stabilization of multiple states *Appl. Phys. Lett.* **80** 3635-3637.
 - [77] Landahl M T 1980 A note on an algebraic instability of inviscid parallel shear flows *J. Fluid Mech.* **98** 243-.
 - [78] Landau L D 1944 On the problem of turbulence *C.R. Acad. Sci. U.R.S.S.* **44** 311-314.
 - [79] Lecoanet D and Kerswell R 2013 Nonlinear optimization of multiple perturbations and stochastic forcing of subcritical ODE systems *Bull. Am. Phys. Soc.* **58** <http://meetings.aps.org/link/BAPS.2013.DFD.L10.4>
 - [80] Lorenz E N 1965 A study of the predicability of a 28-variable atmosphere model *Tellus* **17** 321-333.
 - [81] Luchini P and Bottaro A 1998 Görtler vortices: a backward-in-time approach to the receptivity problem *J. Fluid Mech.* **363** 1-23.
 - [82] Luchini P 2000 Reynolds-number-independent instability of the boundary layer over a flat surface: optimal perturbations *J. Fluid Mech.* **404** 289-309
 - [83] Luchini P and Bottaro A 2014 Adjoint equations in stability analysis *Ann. Rev. Fluid Mech.* **46** 493-517.
 - [84] Machowski J, Bialek J W and Bumby J R 2008 *Power System Dynamics: Stability and Control*, Wiley.
 - [85] Menck P J, Heitzig J, Marwan N and Kurths J 2013 How basin stability complements the linear-stability paradigm *Nature Phys.* **9** 89-92.
 - [86] Monokrousos A, Bottaro A, Brandt L, Di Vita A and Henningson D S 2011 Nonequilibrium thermodynamics and the optimal path to turbulence in shear flows *Phys. Rev. Lett.* **106** 134502.
 - [87] Mu M 2000 Nonlinear singular vectors and nonlinear singular values *Sci. China* **43D** 375-385.
 - [88] Mu M, Duan W S and Wang B 2003 Conditional nonlinear optimal perturbation and its applications *Nonlin. Processes Geophys.* **10** 493-501.
 - [89] Mu M, Sun L and Dijkstra H A 2004 The sensitivity and stability of the ocean's thermohaline circulation to finite-amplitude perturbations *J. Phys. Ocean.* **34** 2305-2315.
 - [90] Mu M and Zhang Z 2006 Conditional nonlinear optimal perturbations of a two-dimensional quasigeostrophic model *J. Atmos. Sci.* **63** 1587-1604.
 - [91] Mu M and Jiang Z N 2008 A new approach to the generation of initial perturbations for ensemble prediction using dynamically conditional perturbations *Chinese Sci. Bull.* **53**, 2062-2068.
 - [92] Mu M and Jiang Z 2008 A method to find perturbations that trigger blocking onset: conditional nonlinear optimal perturbations *Am. Meteor. Soc.* **65** 3935-3946.
 - [93] Mu M, Duan W, Wang Q and Zhang R 2010 An extension of conditional optimal perturbation approach and its applications *Nonlin. Proc. Geophys.* **17** 211-220.
 - [94] Mu M and Jiang Z 2011 Similarities between optimal precursors that trigger the onset of blocking events and optimally growing initial errors in onset prediction *J. Atmos. Sci.* **68** 2860-2877.
 - [95] Mu M, Yu Y S, Xu H and Gong T T 2013 Similarities between optimal precursors for ENSO events and optimally growing initial errors in El Niño predictions *Theor. Appl. Climatol.* **115** 461-469.
 - [96] Mullin T 2011 Experimental Studies of Transition to Turbulence in a Pipe *Ann. Rev. Fluid Mech.* **43** 1-24.
 - [97] Noether F 1921 Das turbulenz problem *Z. angew. Math. Mech.* **1** 125-138.
 - [98] Oortwijn J and Barkmeijer J 1995 Perturbations that optimally trigger weather regimes *J. Atmos. Sci.* **52** 3932-3944.
 - [99] Orr W M 1907 The stability or instability of the steady motions of a perfect fluid and of a viscous fluid. Part I. A perfect fluid. Part II A viscous fluid *Proc. Roy. Irish Acad.* **27** 9-138.
 - [100] Palmer T, Molteni F, Mureau R, Buizza R, Chapelet P and Tribbia J 1993 Ensemble prediction *Proc. ECMWF Seminar* **1** Reading, UK ECMWF 21-66.

- [101] Passaggia P-Y and Ehrenstein U 2013 Adjoint based optimisation and control of a separated boundary-layer flow *Eur. J. Mech. B* **41** 169-177.
- [102] Peixinho J and Mullin T 2007 Finite amplitude thresholds for transition in pipe flow *J. Fluid Mech.* **582** 169-178.
- [103] Pralits J O, Hanifi A and Henningson D S 2002 Adjoint-based optimization of steady suction for disturbance control in incompressible flows *J. Fluid Mech.* **467** 129-161.
- [104] Pringle C C T and Kerswell R R 2010 Using nonlinear transient growth to construct the minimal seed for shear flow turbulence *Phys. Rev. Lett.* **105** 154502.
- [105] Pringle C C T, Willis A P and Kerswell R R 2012 Minimal seeds for shear flow turbulence: using nonlinear transient growth to touch the edge of chaos *J. Fluid Mech.* **702** 415-443.
- [106] Pringle C C T, Willis A P and Kerswell R R 2014 Fully localised nonlinear energy growth optimals in pipe flow *Phys. Fluids* in preparation.
- [107] Rabin S M E A variational approach to determining nonlinear optimal perturbations and minimal seeds *PhD thesis*, University of Cambridge, 2013.
- [108] Rabin S M E, Caulfield C P and Kerswell R R 2012 Variational identification of minimal seeds to trigger transition in plane Couette flow *J. Fluid Mech.* **712** 244-272.
- [109] Rabin S M E, Caulfield C P and Kerswell R R 2014 Designing a more nonlinearly stable laminar flow via boundary manipulation *J. Fluid Mech.* **738** R1.
- [110] Rayleigh Lord 1880 On the stability, or instability, of certain fluid motions *Proc. London Math. Soc.* **11** 57-70.
- [111] Reddy S C and Henningson D S 1993 Energy growth in viscous channel flows *J. Fluid Mech.* **252** 209-238.
- [112] Reddy S C, Schmid P J, Baggett J S and Henningson D S 1998 On stability of streamwise streaks and transition thresholds in plane channel flows *J. Fluid Mech.* **365** 269-303.
- [113] Reshotko E 2001 Transient growth: a factor in bypass transition *Phys. Fluids* **13** 1067-1075.
- [114] Reynolds O 1883a An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous and of the law of resistance in parallel channels *Proc. R. Soc. Lond. Ser. A* **35** 84-99.
- [115] Reynolds O 1883b An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous and of the law of resistance in parallel channels *Phil. Trans. R. Soc. Lond. Ser. A* **174** 935-982.
- [116] Schmid P J 2007 Nonmodal Stability Theory *Ann. Rev. Fluid Mech.* **39** 129-62.
- [117] Schmid P J and Henningson D S 2001 *Stability and Transition in Shear Flows* Springer-Verlag, New York.
- [118] Schmid P J and Brandt L 2014 Analysis of fluid systems: stability, receptivity, sensitivity *Appl. Mech. Rev.* **66** 024803.
- [119] Schneider T M, Eckhardt B and Yorke J A 2007 Turbulence transition and the edge of chaos in pipe flow *Phys. Rev. Lett.* **99** 034502.
- [120] Schneider T M, De Lillo F, Buerhle J, Eckhardt B, Dornemann T, Dornemann K, Freisleben B 2010 Transient turbulence in plane Couette flow *Phys. Rev. E* **81** 015301.
- [121] Schneider T M, Marinc D and Eckhardt B Localized edge states nucleate turbulence in extended plane Couette cells *J. Fluid Mech.* **646**, 441-451.
- [122] Schoppa W and Hussain F 2002 Coherent structure generation in near-wall turbulence *J. Fluid Mech.* **453** 57-108.
- [123] Skufca J D, Yorke J A and Eckhardt B 2006 Edge of chaos in a parallel shear flow *Phys. Rev. Lett.* **96** 174101.
- [124] Sommerfeld A 1908 Ein Beitrag zur hydrodynamischen erklärung der turbulenten Flüssigkeitsbewegungen *Proceedings 4th International Congress of Mathematicians Rome* **III** 116-124.
- [125] Taylor G I Stability of a viscous liquid contained between two rotating cylinders *Phil. Trans. Roy. Soc. Lond.* **223** 289-343.
- [126] Terwisscha van Scheltinga A D and Dijkstra H A 2008 Conditional nonlinear optimal

- perturbations of the double-gyre ocean circulation *Nonlin. Processes Geophys.* **15** 727-734.
- [127] Thomson W (Lord Kelvin) 1887 Stability of fluid motion - rectilinear motion of viscous fluid between two parallel plates *Philos. Mag.* **24** 188-.
- [128] Toth Z and Kalnay E 1993 Ensemble forecasting at NMC: The generation of perturbations *Bull. Amer. Meteor. Soc.* **74**, 2317-2330.
- [129] Toth Z and Kalnay E 1997 Ensemble forecasting at NCEP and the breeding method *Am. Meteor. Soc.* **125** 5297-3319.
- [130] Trefethen L N, Trefethen A E, Reddy S C and Driscoll T A 1993 Hydrodynamic stability without eigenvalues *Science* **261** 578-584.
- [131] Trefethen L N and Embree M 2005 *Spectra and Pseudospectra* Princeton University Press.
- [132] Viswanath D and Cvitanović P 2009 Stable manifolds and the transition to turbulence in pipe flow *J. Fluid Mech.* **627** 215-233.
- [133] Waleffe F 1995 Transition in shear flows: nonlinear normality versus non-normal linearity *Phys. Fluids* **7** 3060-3066.
- [134] Wang Q, Mu M and Dijkstra H A 2012 Application of the conditional nonlinear optimal perturbation method to the predictability study of the Kuroshio Large Meander *Adv. Atmos. Sci.* **29** 118-134.
- [135] Wang Q, Mu M and Dijkstra H A 2013 Effects of nonlinear physical processes on optimal error growth in predictability experiments of the Kuroshio large meander *J. Geophys. Res.* **118** 6425-6436.
- [136] Waugh I C and Juniper M P 2011 Triggering in a thermoacoustic system with stochastic noise *Int. J. Spray and Combustion Dynamics* **3** 225-242.
- [137] Willis A P 2012 On the optimised magnetic dynamo *Phys. Rev. Lett.* **109** 251101.
- [138] Yu YS, Duan WS, Xu H and Mu M 2009 Dynamics of nonlinear error growth and season-dependent predictability of El Niño events in the Zebiak-Cane model *Quart. J. Roy. Meteor. Soc.* **135** 2146-2160.
- [139] Zikanov O Y 1996 On the instability of pipe Poiseuille flow *Phys. Fluids* **8** 2923-2932.
- [140] Ziqing Z U, Mu M and Dijkstra H A 2013 Optimal nonlinear excitation of decadal variability of the North Atlantic thermohaline circulation *Chinese J. Ocean. Limnology* **31** 1356-1362.
- [141] Zuccher S, Luchini P and Bottaro A 2004 Algebraic growth in a Blasius boundary layer: optimal and robust control by mean suction in the nonlinear regime *J. Fluid Mech.* **513** 135-160.
- [142] Zuccher S, Bottaro A and Luchini P 2006 Algebraic growth in a Blasius boundary layer: Nonlinear optimal disturbances *Eur. J. Mech. B* **25** 1-17.