Dyadic Approximation in the Middle-Third Cantor Set

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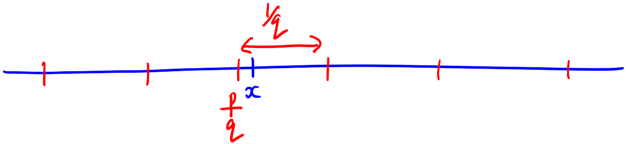
University of Bristol

26th May 2021, One-Day Ergodic Theory Meeting (online)

Joint work with Sam Chow (Warwick) and Han Yu (Cambridge).

Diophantine Approximation



Fact If $x \in \mathbb{R}$, and $q \in \mathbb{N}$, $\exists\, p \in \mathbb{Z}$ s.t.

$$\left| x - \frac{p}{q} \right| < \frac{1}{q}.$$

Theorem (Dirichlet, 1842) For any $x \in \mathbb{R}$, $\exists$ infinitely many $q \in \mathbb{N}$ s.t.

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$$

for some $p \in \mathbb{Z}$.

Khintchine's Theorem

Given $\psi : \mathbb{N} \to [0, \infty)$, we define the ψ-*well approximable points* as

$$\mathcal{A}(\psi) := \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

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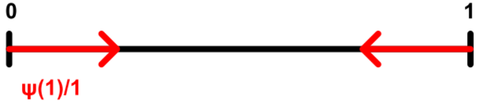

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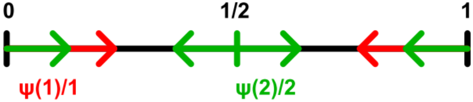

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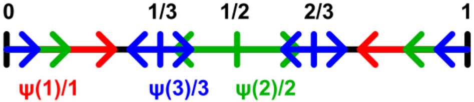

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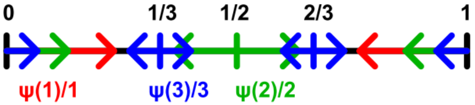

Khintchine's Theorem (1924)

For any monotonic approximating function $\psi : \mathbb{N} \to [0, \infty)$,

$$\mathcal{L}(\mathcal{A}(\psi)) = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty} \psi(q) < \infty \,, \\ 1 & \text{if} \quad \sum_{q=1}^{\infty} \psi(q) = \infty \,. \end{cases}$$

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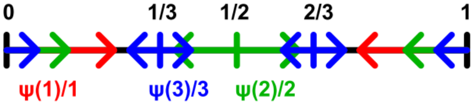

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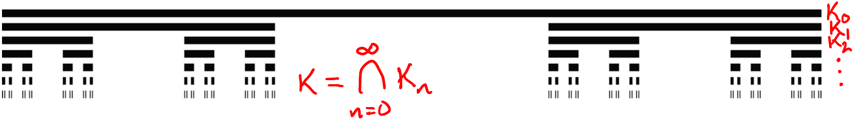
Question

If $X \subset \mathbb{R}$, what can we say about $X \cap \mathcal{A}(\psi)$?

Question (Mahler, 1984)

How close can irrational elements of Cantor's set be approximated
(i) by rational numbers in Cantor's set, and
(ii) by rational numbers not in Cantor's set?



Diophantine Approximation on Fractals: Mahler's Question

Question (Mahler, 1984)

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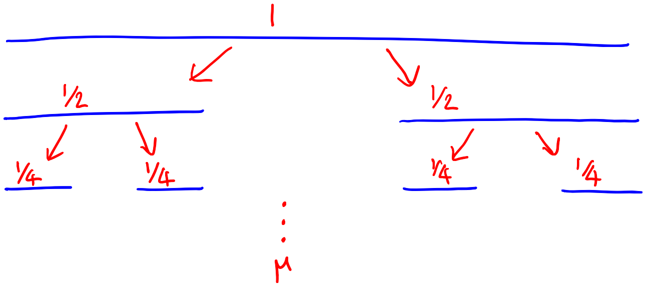
(ii) by rational numbers not in Cantor's set?

Many authors, many works: Weiss (2001), Kleinbock–Lindenstrauss–Weiss (2004), Kristensen (2006), Levesley–Salp–Velani (2007), Bugeaud (2008), Bugeaud–Durand (2016), Fishman + various collaborators (2011–2018), Khalil–Lüthi (2021+), Yu (2021+), . . .

Triadic Approximation in the Middle-Third Cantor Set

- K — middle-third Cantor set.
- $\gamma := \dim_H K = \frac{\log 2}{\log 3}$.
- μ — natural probability measure on K (i.e. $\mu := \mathcal{H}^{\gamma}|_K$).
- Note μ is γ-Ahlfors regular; i.e. $\mu(B(x,r)) \asymp r^{\gamma}$ for $x \in K$ and $r > 0$.



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- $\mathcal{B} := \{3^n : n = 0, 1, 2, \dots\}$.
- Given $\psi : \mathbb{N} \to [0, \infty)$, let

$$\mathcal{A}_{\mathcal{B}}(\psi) := \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathcal{B} \right\}.$$

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Theorem (Levesley – Salp – Velani, 2007)

For $\psi : \mathbb{N} \to [0, \infty)$,

$\mathcal{A}_{\mathcal{B}}(\psi) \cap K$

$$\mu(\mathcal{A}_{\mathcal{B}}(\psi)) = \begin{cases} 0 & \text{if} \quad \sum_{n=1}^\infty \psi(3^n)^\gamma < \infty, \\ 1 & \text{if} \quad \sum_{n=1}^\infty \psi(3^n)^\gamma = \infty. \end{cases}$$

Dyadic Approximation in the Middle-Third Cantor Set

Question

What about dyadic approximation in the Cantor set, i.e. what if $\mathcal{B} = \{2^n : n = 0, 1, 2, \dots\}$*?*

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Conjecture (Velani)

For $\psi : \mathbb{N} \to [0, \infty)$, if $\mathcal{B} = \{2^n : n = 0, 1, 2, \dots\}$,

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- Let $A_n = \bigcup_{a=0}^{2^n} B\left(\frac{a}{2^n}, \frac{\psi(2^n)}{2^n}\right)$ for $n \in \mathbb{N}$.
- Note that $A_{\mathcal{B}}(\psi) = \limsup_{n \to \infty} A_n = \bigcap_{j=0}^{\infty} \bigcup_{n=j}^{\infty} A_n$.

$\sum \mu(A_n) < \infty$

- Fix n and suppose that $\boxed{\frac{\psi(2^n)}{2^n} \approx 3^{-N}}$.
- Suppose that the dyadic rationals are "uniformly distributed" in $[0,1]$.

Heuristics

- We expect to find $\approx \frac{2^N}{3^N} \times 2^n$ dyadic rationals with denominator 2^n "near" K_N.

$$\Rightarrow \mu(A_n) \ll \frac{2^N}{3^N} \times 2^n \times \left(\frac{\psi(2^n)}{2^n}\right)^\gamma \qquad \frac{\psi(2^n)}{2^n} \approx 3^{-N}$$

$$\approx 2^N \times \left(\frac{\psi(2^n)}{2^n}\right) \times 2^n \times (3^{-N})^\gamma$$

$$= \psi(2^n).$$

- By the First Borel-Cantelli Lemma, if $\sum_{n=0}^{\infty} \psi(2^n) < \infty$, then $\mu(W_B(\psi)) = 0$.

Our Results

Proposition (Benchmark Convergence Result, A. – Chow – Yu)

For $\psi : \mathbb{N} \to [0, \infty)$, *if* $\displaystyle\sum_{n=1}^{\infty} \psi(2^n)^\gamma < \infty$, *then* $\mu(\mathcal{A}_{\mathcal{B}}(\psi)) = 0$.

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Theorem (Main Convergence Theorem, A. – Chow – Yu)

If $\sum_{n=1}^{\infty} (2^{-\log n/(\log\log n \cdot \log\log\log n)} \psi(2^n)^{\gamma} + \psi(2^n)) < \infty$, *then* $\mu(\mathcal{A}_{\mathcal{B}}(\psi)) = 0$.

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Theorem (Main Divergence Theorem, A. – Chow – Yu)

For $\psi(2^n) = 2^{-\log\log n/\log\log\log n}$, *we have* $\mu(\mathcal{A}_{\mathcal{B}}(\psi)) = 1$.

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Main ideas: Fourier Analysis and bounds on digit changes. $2^n m$

Digit Changes in Base 2 and Base 3

For $n \in \mathbb{N}$, let $D_2(n)$ denote the number of *digit changes* in the base 2 expansion of n. Likewise, let $D_3(n)$ denote the number of digit changes in the base 3 expansion of n.

$n = 20$

base 2: 10100 $\quad D_2(20) = 3$

base 3: 202 $\quad D_3(20) = 2.$

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Lemma (Stewart (1980), Bugeaud – Cipu – Mignotte (2013))

For sufficiently large $n \in \mathbb{N}$, we have

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If we could improve the bound in the above lemma, we could strengthen our earlier results.

Question Is $D_2(n) + D_3(n) \asymp \log n$ for large enough n?

Relatively straightforward to show $D_2(n) + D_3(n) \ll \log n$.

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If we could improve the bound in the above lemma, we could strengthen our earlier results. In particular, a bound of $D_2(n) + D_3(n) \gg \log \log n$ would imply the convergence part of Velani's Conjecture.

Conditional Convergence Results

Theorem (A. – Chow – Yu)

Assuming the Lang–Waldschmidt Conjecture, for sufficiently large $y \in \mathbb{N}$*, we have*

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Assuming the Lang–Waldschmidt Conjecture, for sufficiently large $y \in \mathbb{N}$*, we have*

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Corollary

Conditional on the Lang–Waldschmidt Conjecture, the convergence part of Velani's Conjecture is true. Namely; if

$$\sum_{n=1}^{\infty} \psi(2^n) < \infty,$$

then $\mu(\mathcal{A}_{\mathcal{B}}(\psi)) = 0$.

Conditional Divergence Results

Theorem (A. – Chow – Yu)

Suppose $D_2(y) + D_3(y) \geq h(y)$ *for all* $y \geq 1$, *where* $h : \mathbb{N} \to (0, \infty)$ *is an increasing function.*

1. *If* $h(y) \gg \log y$, *then for* $\psi(2^n) = \frac{1}{n}$ *we have* $\mu(\mathcal{A}_{\mathcal{B}}(\psi)) = 1$.
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`Thank you for your attention!`