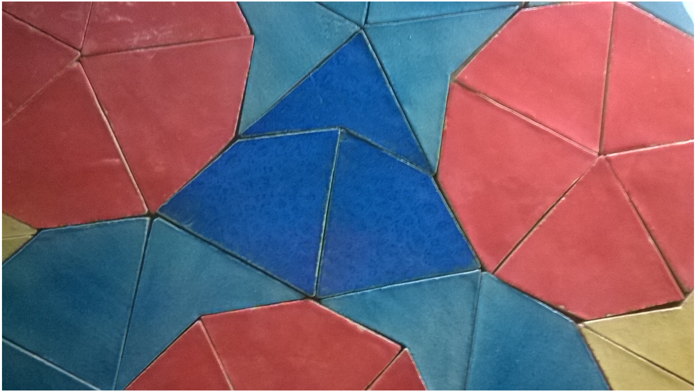
Cut and project sets and Diophantine approximation

Henna Koivusalo
based on joint works with Alan Haynes, Antoine Julien, Lorenzo Sadun, Jamie Walton

University of York

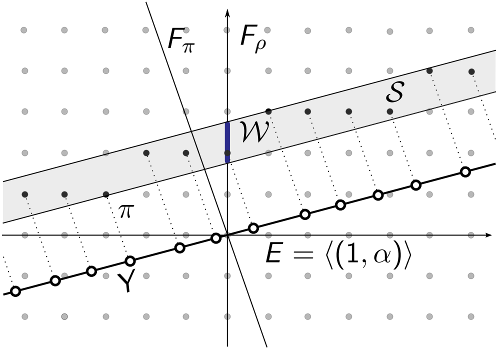
March 2016

Tilings



Cut and project sets

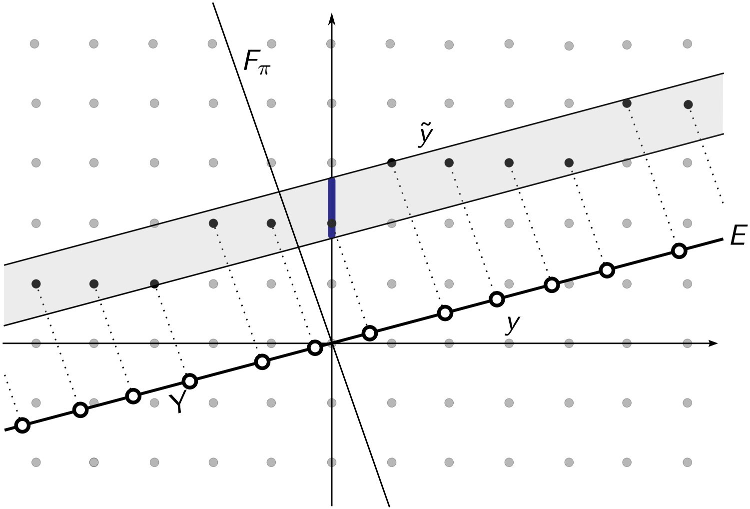
Cut and project sets



$$Y = Y_{\mathcal{W}} = \pi(\mathcal{S} \cap \mathbb{Z}^k) \subset E$$

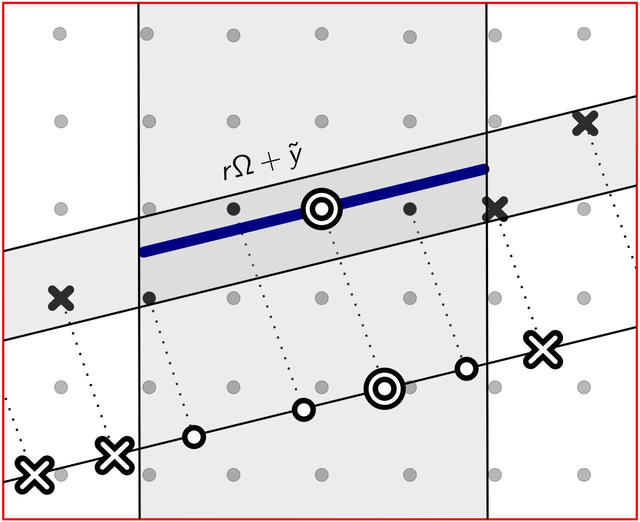
- subspace E totally irrational
- F_π transversal to E
- $F_\rho = \mathbb{R}^{k-d}$
- $\pi : \mathbb{R}^k \to E$ projection along F_π
- $\mathcal{W} \subset F$ window
- $\mathcal{S} = E + \mathcal{W}$ strip
- $E =$ graph of a linear $\alpha : \mathbb{R}^d \to \mathbb{R}^{k-d}$

Repetition of patterns



Patterns in Y

Patch of size r around a point $y \in Y$?



$$P(y, r) = \{y' \in Y \mid \rho(\tilde{y}' - \tilde{y}) \in r\Omega\}$$

Repetition of patterns

- $P(y, r)$ equivalent to $P(y', r)$ if $P(y, r) = P(y', r) + y - y'$
- $[P(y, r)] = \mathcal{P}(y, r)$

Repetition of patterns

- $P(y, r)$ equivalent to $P(y', r)$ if $P(y, r) = P(y', r) + y - y'$
- $[P(y, r)] = \mathcal{P}(y, r)$

Some previous work

- $r^d \leq$ number of different patches of size $r \leq r^{d(k-d)}$(Julien 2010)
- linear repetition of patches: Lagarias and Pleasants 2003, Besbes, Boshernitzan and Lenz 2013

We answer the questions:

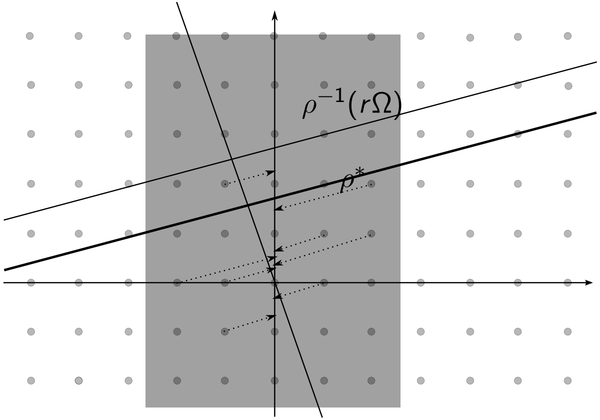
- How long do we have to wait to see all patches of size r?
- How many different frequencies do patches of size r have?

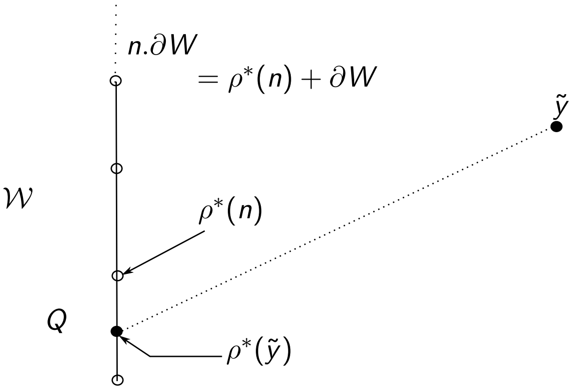
We answer the questions:

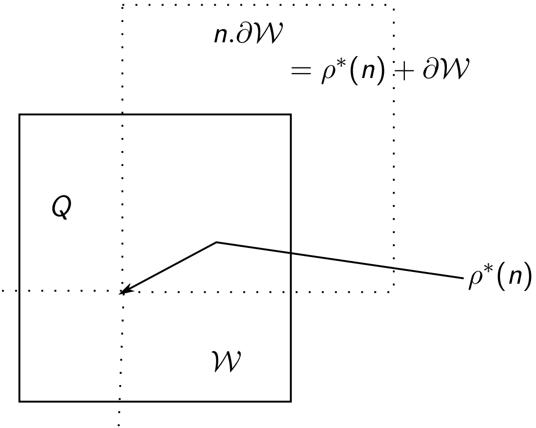
- How long do we have to wait to see all patches of size r?
- How many different frequencies do patches of size r have?

... and find an equivalent form of a famous open problem in Diophantine approximation.

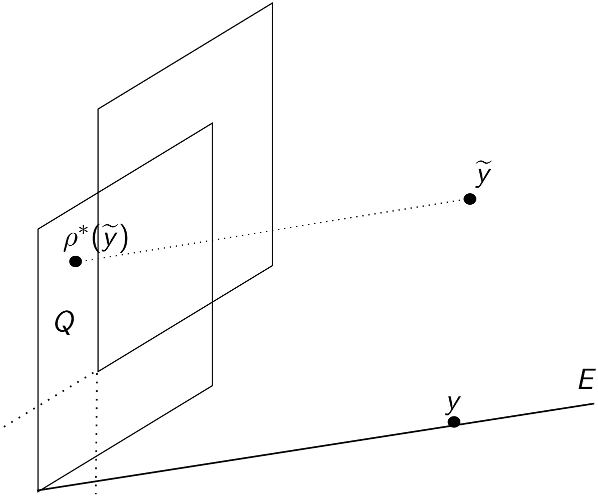
An observation


$\rho^{-1}(r\Omega)$
ρ^*


$n.\partial W$
$= \rho^*(n) + \partial W$
$\tilde{y}$
$\mathcal{W}$
$\rho^*(n)$
Q
$\rho^*(\tilde{y})$



$n.\partial\mathcal{W}$
$= \rho^*(n) + \partial\mathcal{W}$
Q
$\rho^*(n)$
$\mathcal{W}$


$\widetilde{y}$
$\rho^*(\widetilde{y})$
Q
E
y

An observation

Lemma (Haynes, K., Sadun, Walton)

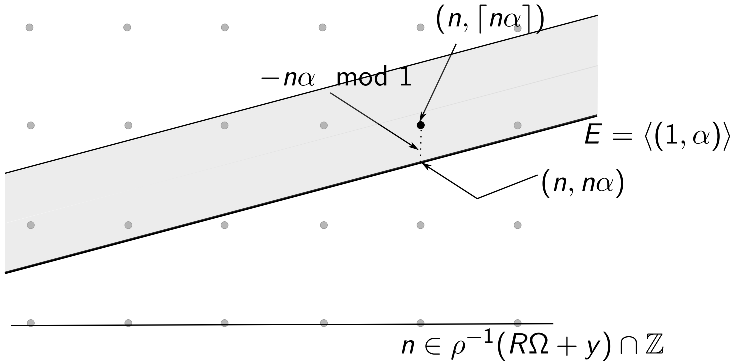
The action of the points $\rho^{-1}(r\Omega) \cap \mathbb{Z}^k$ *by* $n.x = \rho^*(n) + x$*, on the boundary of* $\mathcal{W}$ *partitions the window* $\mathcal{W}$*. Connected components in this partition correspond exactly to patches of size* r *in* Y*, that is, for every* $\mathcal{P} = \mathcal{P}(y, r)$ *there is a connected component* Q *such that* $P(y', r) \in \mathcal{P}$ *iff* $\rho^*(\tilde{y}), \rho^*(\tilde{y}') \in Q$.

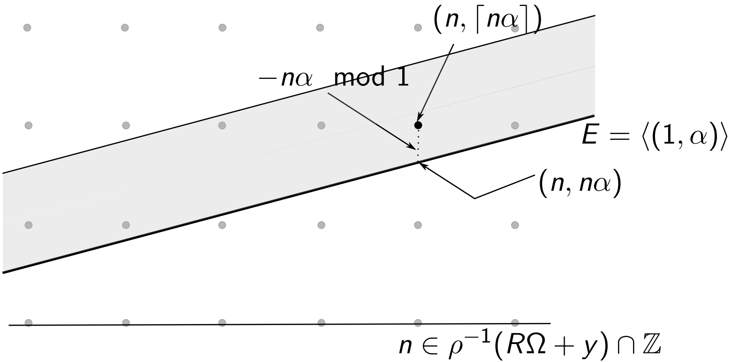
An observation

Lemma (Haynes, K., Sadun, Walton)

The action of the points $\rho^{-1}(r\Omega) \cap \mathbb{Z}^k$ by $n.x = \rho^*(n) + x$, on the boundary of $\mathcal{W}$ partitions the window $\mathcal{W}$. Connected components in this partition correspond exactly to patches of size r in Y, that is, for every $\mathcal{P} = \mathcal{P}(y, r)$ there is a connected component Q such that $P(y', r) \in \mathcal{P}$ iff $\rho^*(\tilde{y}), \rho^*(\tilde{y}') \in Q$.

Similar observations: Berthe and Vuillon 2000, Julien 2010


$(n, \lceil n\alpha \rceil)$
$-n\alpha \mod 1$
$E = \langle (1, \alpha) \rangle$
$(n, n\alpha)$
$n \in \rho^{-1}(R\Omega + y) \cap \mathbb{Z}$



$(\alpha(n) \mod 1)_{n \in \rho^{-1}(R\Omega + y) \cap \mathbb{Z}^d}$ visits every connected component

$$\iff$$

$P(y, R)$ contains all r-patches for all y

Our results

Some more notation

- **frequency**

$$\xi_{\mathcal{P}(y,r)} = \lim_{R\to\infty} \frac{\#\{y' \in Y \mid |y'| \leq R, y' \text{ equivalent to } y\}}{\#\{y' \in Y \mid |y'| \leq R\}}$$

- **frequency spectrum**

$$\xi(r) = \text{ distinct values of } \xi_{\mathcal{P}}$$

- we call $R : \mathbb{R}_+ \to \mathbb{R}_+$ a **repetitivity function** if any patch of size $R(r)$ contains all patches of size r

Our results: Frequencies of patches

Theorem (Haynes, K., Walton, Sadun 2014)

Assume $\mathcal{W}$ is a parallelotope with vertices in $\mathbb{Z}^{k-d} \cap F$, and Y aperiodic and totally irrational. Then

- *for a full dimensional set of choices for E, there is $C > 0$ such that $\#\xi(r) \leq C$ for all $r > 0$,*
- *for a full measure set of choices for E, for any $\epsilon > 0$ and all large r, $\#\xi(r) \leq (\log r)^{(1+\epsilon)(d+1)(k-d)}$*
- *for a positive dimensional set of choices for E, we can choose Ω to be a polytope and find $\epsilon > 0$ in such a way that*

$$\limsup_{r\to\infty} \frac{\#\xi(r)}{r^{\epsilon}} = \infty.$$

Our results: Repetitivity

Theorem (Haynes, K., Walton 2015)

Let Y as above be defined by the linear form $\alpha = (\alpha_1, \ldots, \alpha_{k-d})$. When $\mathcal{W}$ is the unit square, Y has a linear repetitivity function if and only if the sum of the ranks of the kernels of $\alpha_i \mod 1$ is $d(k-d-1)$ and each α_i is **relatively badly approximable**.

Theorem (Haynes, Julien, K., Walton 2016)

For $\epsilon > 0$, for almost all choices of E, the growth rate of the repetitivity function has the upper bound $r^{k-d}(\log r)^{(2k-1)/d-1+\epsilon}$. In the case $k-d=1$, it also has a lower bound $r(\log r)^{1/d}$.

Theorem (Haynes, K., Walton 2015)

The Littlewood conjecture holds if and only if **super perfectly ordered** *cut and project sets do not exist.*

Thank you!