Horocycle flow orbits of translation surfaces

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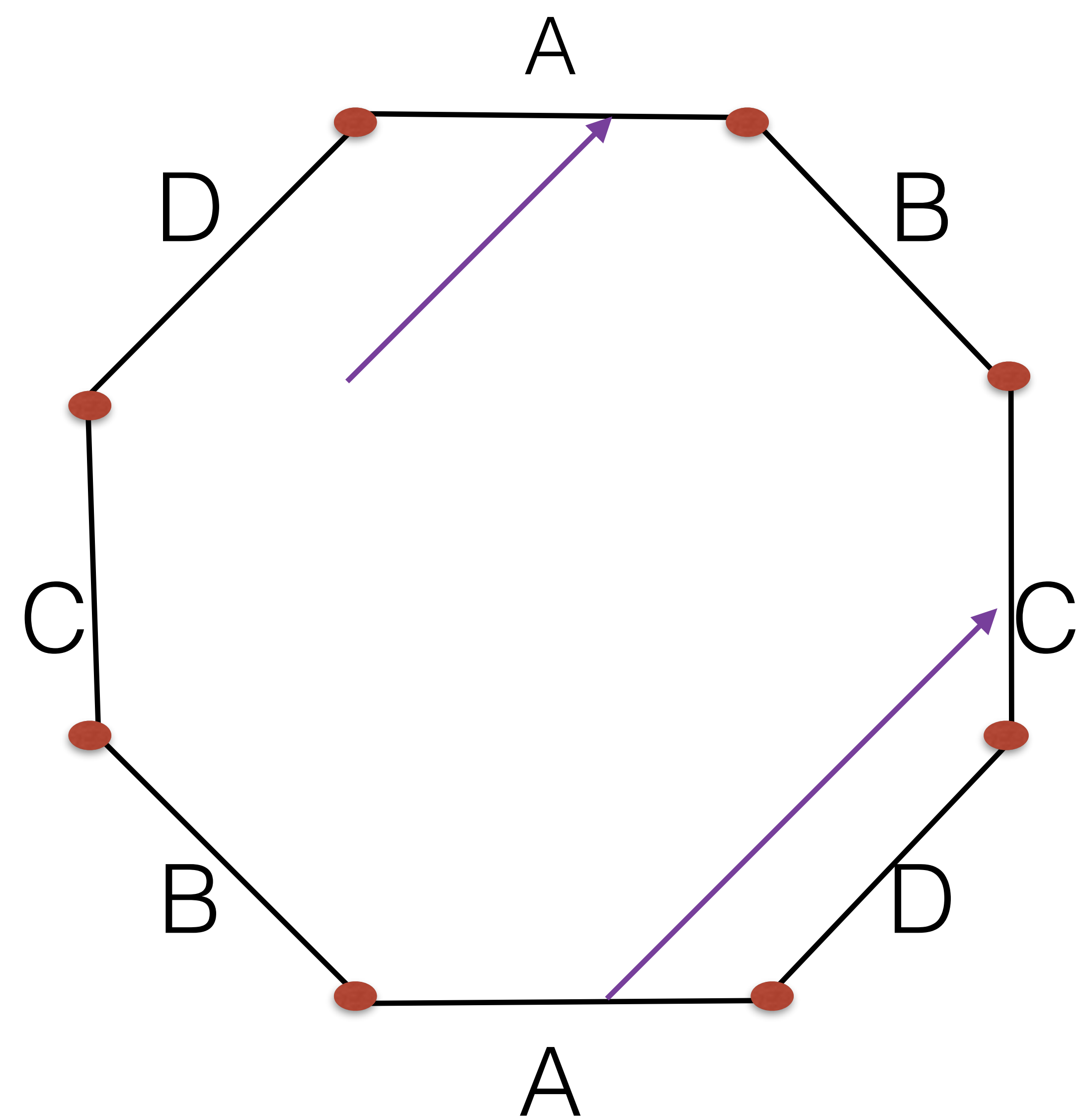
A translation surface is a closed, compact, 2-real-dimensional manifold M together with a finite subset of points P⊂M and a maximal atlas of charts such that the restriction to M\P of each transition map is a translation.



⇒ On M\P, “directions” are well-defined
e.g. positive vertical direction

⇒ Euclidean metric

Points in P are called cone points.

⇒ The cone angle of a cone point is of the form $2\pi(k+1)$, $k \in \mathbb{N}^{\geq 0}$

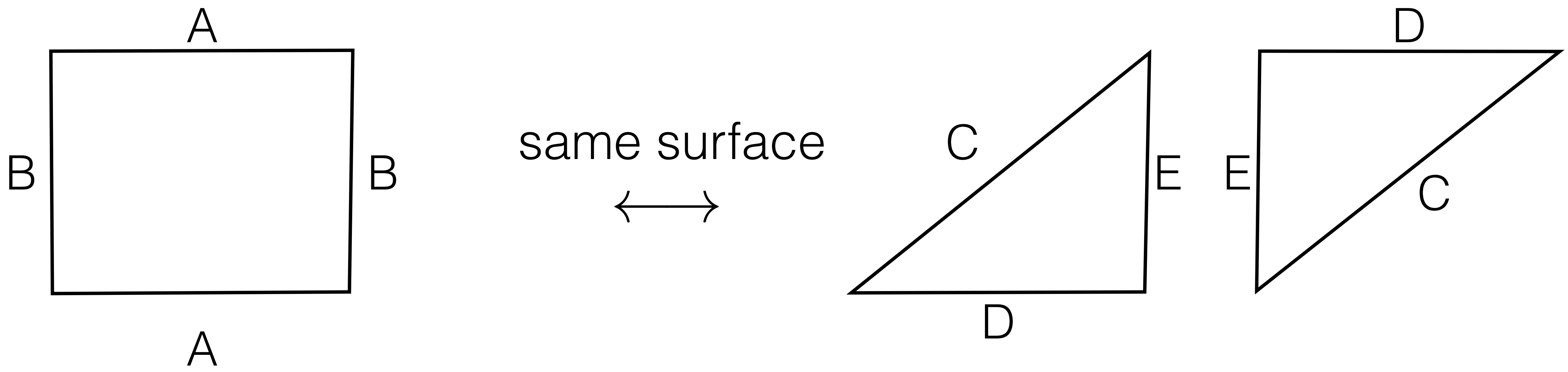
k is the order of the cone point.

$$\Rightarrow \sum_{p \in P} \text{order}(p) = 2g - 2$$

P = {●}

● has cone angle 6π,
i.e. order 2

We care about underlying surface, not specific representation in terms of polygons/charts. i.e:



$\mathcal{H}(k_1, \ldots, k_n)$ consists of all translation surfaces whose cone points are of orders $k_1, \ldots, k_n$.

A"stratum in moduli space of translation surfaces"

Topology on $\mathcal{H}(k_1, \ldots, k_n)$: slightly perturbing the polygon edges yields "nearby" surfaces.

$SL_2(\mathbb{R})$ acts on $\mathcal{H}(k_1, \ldots, k_n)$

(Each element determines a linear transformation of the plane; apply this transformation to the polygons/charts.)

Action of one-parameter subgroups:

geodesic flow, G, action of matrices of form $g_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$

horocycle flow, H, action of matrices of form $h_t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

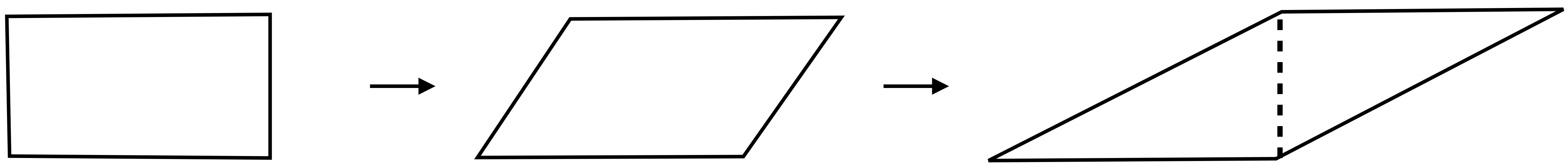
rotations, actions of matrices of form $r_\theta := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin\theta & \cos(\theta) \end{pmatrix}$

Understanding the dynamics of the SL2R action has been and continues to be the focus of a great deal of work by many people.

Dynamics of horocycle flow?

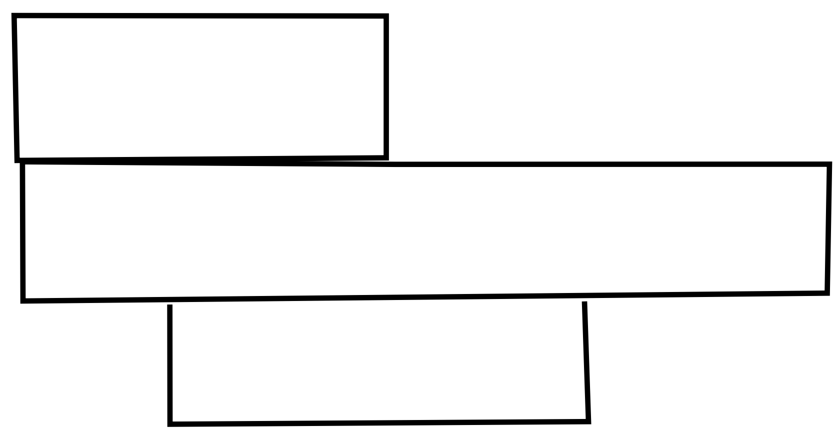
Notation: $\overline{H \cdot S}$ means closure in the stratum of the orbit of S under H

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ty \\ y \end{pmatrix}$$



t=0

period: t=modulus (=height/width)



periodic (under H) if moduli all rationally related

Smillie and Weiss (2004):
Minimal sets for H <—> orbit closures (under H) surfaces that admit a horizontal cylinder decomposition.

(A minimal subset for a group G acting on a space X is a closed, G-invariant set that is a minimal such set with respect to inclusion.)

Joint with **Jon Chaika**:

Theorem 1: For any translation surface M,

$$\overline{H \cdot r_\theta \cdot M} = \overline{SL_2(\mathbb{R}) \cdot M}$$

for (Lebesgue) almost every angle $\theta \in S^1$.

Joint with **Jon Chaika** (2015):

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Theorem 2: The following are equivalent:

(i) M is a lattice (Veech) surface.
(ii) For every angle $\theta \in S^1$, every H-minimal subset of $\overline{H r_\theta \cdot M}$ is a periodic H-orbit.
(iii) Every H-minimal subset of $\overline{SL_2(\mathbb{R}) \cdot M}$ is a periodic H-orbit.

(A lattice surface is a surface whose SL2R orbit is closed.)

Theorem 1: For any translation surface M, $\overline{H \cdot r_\theta \cdot M} = \overline{SL_2(\mathbb{R}) \cdot M}$ for (Lebesgue) almost every angle $\theta \in S^1$.

Is the "almost every" necessary?

An easy way to see the answer is "yes": saddle connection directions. Horocycle flow preserves horizontal segments.

So if $r_\theta \cdot M$ has a horizontal saddle connection, every surface in $Hr_\theta \cdot M$ will have a horizontal saddle connection of the same length.

Open question: for what angles θ is $\overline{H \cdot r_\theta \cdot M} \neq \overline{SL_2(\mathbb{R}) \cdot M}$?

Eskin-Mirzakhani-Mohammadi (2013): $SL_2(\mathbb{R})$ orbits are "nice."

- technical term "affine invariant submanifolds"
- determined by linear equations in period coordinates
- no boundary
- SL2R invariant, ergodic probability measure with full support

Bainbridge-Smillie-Weiss (2016): classified H-inv. prob. measures in $\mathcal{H}(1, 1)$
Smillie-Weiss: examples of H-orbit closures that are very far from "nice."

Sketch of proof of Theorem 1

Fix an SL2R orbit closure $\mathcal{M}$ with associated SL2R inv. ergodic prob. measure μ. Fix $M \in \mathcal{M}, \overline{SL_2(\mathbb{R}) \cdot M} = \mathcal{M}$.

(Goal: for $\epsilon > 0$, show $Hr_\theta \cdot M$ is ϵ-dense in $\mathcal{M} \cap K_\epsilon$ for a full measure set of θ.)

The Mautner phenomenon for SL2R:

An SL2R invariant measure on $\mathcal{H}$ is ergodic for SL2R if and only if it is ergodic for H.

So μ is ergodic for H.

So μ -almost every surface in $\mathcal{M}$ has a dense H-orbit.

Def: Say M is (L,ϵ)-nice if $\bigcup_{s=0}^{L} h_s \cdot M$ is ϵ-dense in $\mathcal{M} \cap K_\epsilon$

So there exist sets $U_{L,\epsilon}$ of (L,ϵ)-nice surfaces of measure arbitrarily close to 1.

(New goal: show $Hr_\theta \cdot M$ hits $U_{L,\epsilon}$ for a full measure set of θ)

A result from EMM guarantees that there exist arbitrarily large t s.t. all but percent of the "circle" (in theta) $g_t r_\theta \cdot M$ is in $U_{L,\epsilon}$.

So for arbitrarily large t, $g_t r_\theta \cdot M$ contains a point u in $U_{L,\epsilon}$.

But for very large t, an ϵ-arc of $g_t r_\theta \cdot M$ ϵ-approximates a length L segment of the horocycle flow applied to u.

So for sufficiently large t, ϵ -arcs of $g_t r_\theta \cdot M$ are 2epsilon-dense in $\mathcal{M} \cap K_\epsilon$.

Cartan decomposition: every element of SL2R can be written as $r_a g_b r_c$ for some a, b, c.

Use Cartan decomposition of h_s to turn into a statement that for sufficiently large s, any ϵ-arc of $h_s r_\theta \cdot M$ is 2epsilon-dense.

Push this train of logic more…take intersections of sets as $\epsilon \to 0$ …
… and this leads to Theorem 1.

$$\overline{H \cdot r_\theta \cdot M} = \overline{SL_2(\mathbb{R}) \cdot M} \quad \text{for almost every angle } \theta \in S^1.$$

Thank you!