Bernoulli convolutions

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Bernoulli convolutions

Fix a number $0 < \lambda < 1$ and denote by μ_λ the law of the random variable

$$\sum_{n=0}^{\infty} A_n \lambda^n,$$

where A_n are independent random variables with law

$$\mathbf{P}(A_n = 1) = \mathbf{P}(A_n = -1) = 1/2.$$

- $\lambda < 1/2$: μ_λ is singular supported on a Cantor set (Devil's staircase).
- $\lambda = 1/2$: μ_λ is the normalized Lebesgue on $[-2, 2]$.
- $\lambda > 1/2$: μ_λ may be singular or a.c.

Theorem (Erdős '39)

If λ^{-1} is a Pisot number, e.g.

$$\lambda = \frac{\sqrt{5}-1}{2},$$

then μ_λ is singular.

Theorem (Erdős '40)

There is a number $c > 0$, such that μ_λ is a.c. for almost all $\lambda \in [1-c, 1]$.

Theorem (Solomyak '95)

For almost all $\lambda \in [1/2, 1]$, μ_λ *is a.c.*

Theorem (Hochman '14)

The set

$$\{\lambda \in [1/2, 1] : \dim \mu_\lambda < 1\}$$

is of dimension 0.

Theorem (Shmerkin '14)

The set

$$\{\lambda \in [1/2, 1] : \mu_\lambda \text{ is singular}\}$$

is of dimension 0.

Theorem (Hochman '14)

If $1/2 < \lambda < 1$ is algebraic, then

$$\dim \mu_\lambda = \min\left\{1, \frac{h_\lambda}{-\log \lambda}\right\},$$

where

$$h_\lambda = \lim_{N\to\infty} \frac{1}{N} H(A_0\lambda^0 + \ldots + A_{N-1}\lambda^{n-1}),$$

and $H(\cdot)$ denotes Shannon entropy.

Theorem (Breuillard,V '15)

If λ is an algebraic number, then

$$0.4 \cdot \min\{1, \log M_\lambda\} \leq h_\lambda \leq \min\{1, \log M_\lambda\},$$

where M_λ is the Mahler measure of λ.

- In this presentation $\log$ is base 2.
- For an algebraic number λ with minimal polynomial

$$a_n(z-\lambda_1)\cdots(z-\lambda_d)$$

the Mahler measure is defined as

$$M_\lambda = a_n \prod_{|\lambda_i|>1} |\lambda_i|.$$

Corollary (Breuillard, V '15)

If $\lambda \geq \min\{2, M(\lambda)\}^{-0.4}$, *then* $\dim \mu_\lambda = 1$.

- In this presentation $\log$ is base 2.
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the Mahler measure is defined as

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Theorem (V '16)

There is an absolute constant $c>0$, such that if $\lambda<1$ is an algebraic number that satisfies

$$\lambda > 1 - c\min\{\log M_\lambda, (\log M_\lambda)^{-1.0001}\},$$

then μ_λ is absolutely continuous.

Let X be a random variable and $s > 0$ a number. The entropy of X at scale s is:

$$H(X; s) := \mathbf{E}_t(H(\lfloor s^{-1}X + t \rfloor)).$$

Furthermore, we also consider the following notion of conditional entropy between two scales:

$$H(X; s_1|s_2) := H(X; s_1) - H(X; s_2).$$

To show $\dim \mu_\lambda = 1$, we need:

$$H(\mu_\lambda; 2^{-n}|2^{-n+1}) \to 1.$$

To show μ_λ is a.c., we need e.g.:

$$H(\mu_\lambda; 2^{-n}|2^{-n+1}) = 1 - O(n^{-2}).$$

Idea: Write μ_λ as a convolution of measures:

$$\mu_\lambda = \nu_1 * \cdots * \nu_K$$

such that for each i we have some preliminary lower bounds for

$$H(\nu_1; s_1|s_2)$$

for suitably chosen scales $s_1 < s_2$.
Then try to show that these bounds improve, when we take convolutions.

Theorem (V '16)

For every $\alpha > 0$, there are numbers $C, c > 0$ such that the following holds. Let μ, ν be two compactly supported probability measures on $\mathbf{R}$. Let $\sigma_2 < \sigma_1 < 0$ and $\beta > 0$ be real numbers. Suppose that

$$H(\mu; 2^{\sigma}|2^{\sigma+1}) < 1 - \alpha$$

for all $\sigma_2 < \sigma < \sigma_1$. Suppose further that

$$H(\nu; 2^{\sigma_2}|2^{\sigma_1}) > \beta(\sigma_1 - \sigma_2).$$

Then

$$H(\mu * \nu; 2^{\sigma_2}|2^{\sigma_1}) > H(\mu; 2^{\sigma_2}|2^{\sigma_1}) + c\beta(\log \beta^{-1})^{-1}(\sigma_1 - \sigma_2) - C.$$

Notation: For $I \subset (0,1]$ write μ_λ^I for the law of the random variable

$$\sum_{n:\lambda^n \in I} A_n \lambda^n.$$

For simplicity, assume that $\lambda = p/q$ is rational. Then any two points in the support of $\mu_\lambda^{(\lambda^n,1]}$, are of distance at least q^{-n}. Then

$$H(\mu_\lambda^{(\lambda^n,1]}; q^{-n}) = n,$$

and

$$H(\mu_\lambda^{(\lambda^n,1]}; q^{-n} | 2^{-n/2}) \geq n/2.$$

Similarly

$$H(\mu_\lambda^{(\lambda^{n(k+1)},\lambda^{nk}]}; q^{-n}\lambda^{nk} | 2^{-n/2}\lambda^{nk}) \geq n/2.$$

If λ is close to 1 so that $\lambda^{nk} > 2^{-n/4}$, we get

$$H(\mu_\lambda^{(\lambda^{n(k+1)},\lambda^{nk}]}; q^{-n}|2^{-n/2}) \geq n/4.$$

Observe:

$$\mu_\lambda^{(\lambda^{n(k+1)},1]} = \mu_\lambda^{(\lambda^{n(k+1)},\lambda^{nk}]} * \mu_\lambda^{(\lambda^{nk},1]}.$$

Apply the theorem on entropy increase:

- either $H(\mu_\lambda^{(\lambda^{nk},1]}; r|2r) > 1-\alpha$ for some $q^{-n} < r < 2^{-n/2}$,
- or $H(\mu_\lambda^{(\lambda^{n(k+1)},1]}); q^{-n}|2^{-n/2}$ is much larger than $H(\mu_\lambda^{(\lambda^{nk},1]}); q^{-n}|2^{-n/2}$.

The second alternative cannot hold forever, hence we must have

$$H(\mu_\lambda^{(r,1]}; r^2|2r^2) \geq 1-\alpha$$

for some r.

Theorem (V '16)

There is an absolute constant $C > 0$ *such that the following holds. Let* $\mu, \widetilde{\mu}$ *be two compactly supported probability measures on* $\mathbf{R}$ *and let* $\alpha, r > 0$ *be real numbers. Suppose that*

$$H(\mu; s|2s) \geq 1 - \alpha \quad \textit{and} \quad H(\widetilde{\mu}; s|2s) \geq 1 - \alpha$$

for all s *with* $|\log r - \log s| < C \log \alpha^{-1}$.
Then

$$H(\mu * \widetilde{\mu}; r|2r) \geq 1 - C(\log \alpha^{-1})^3 \alpha^2.$$

Thank You!