A Selective Survey of Projections

A talk to mark the 60th anniversary of John Marstrand's fundamental paper

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SOME FUNDAMENTAL GEOMETRICAL PROPERTIES OF PLANE SETS OF FRACTIONAL DIMENSIONS

By J. M. MARSTRAND

[Received 27 March 1953.—Read 16 April 1953]

1. Introduction

1.1. Notation and Definitions. Given any positive number q, by $A(q)$ we denote any sequence of convex areas U, each of diameter $dU < q$.

Suppose that we have a plane set of points E, and that $A(E, q)$ denotes any set $A(q)$ which contains E. Then by

$$\Lambda_q^s E$$

we denote the lower bound of

$$\sum_{A(E,q)} (dU)^s$$

taken over all possible sets $A(E, q)$.

The *outer Hausdorff s-dimensional measure* of E is given by

$$\Lambda^s E = \lim_{q \to 0} \Lambda_q^s E.$$

Proceedings of the London Mathematical Society(3),**4** (1954), 257-302

Hausdorff measure and dimension

Let $E \subseteq \mathbb{R}^n$. (Note: throughout this talk we will assume all such sets are 'reasonable', i.e. Borel or analytic.) For $s > 0, \delta > 0$ let

$$\mathcal{H}^s_\delta(E) = \inf \Big\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s : E \subseteq \bigcup_{i=1}^{\infty} U_i,\ \operatorname{diam} U_i \leq \delta \Big\}.$$

Then

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E)$$

is the s-dimensional Hausdorff measure of E.

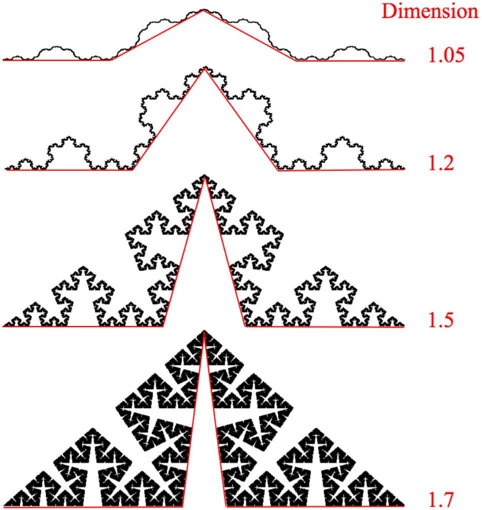
The Hausdorff dimension of $E \subset \mathbb{R}^n$ is

$$\dim_H E = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = \infty\}.$$

If you prefer:

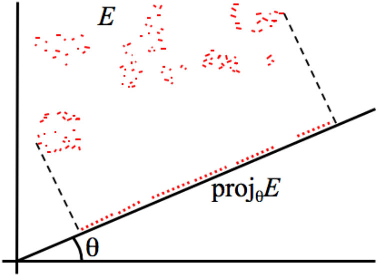
$$\dim_H E = \inf \Big\{ s : \text{for all } \epsilon > 0 \text{ there is a countable cover } \{U_i\} \text{ of } E \text{ such that } \sum_i (\operatorname{diam} U_i)^s < \epsilon \Big\}.$$

Von Koch curves of various dimensions



John Marstrand's 1954 paper

- The paper has been highly cited in recent years, usually for the 'projection theorems' in the plane:

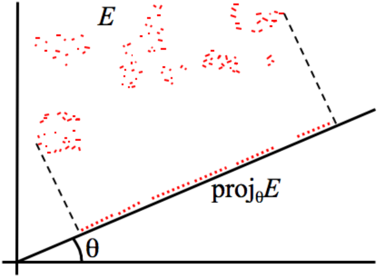


Theorem I *Any s-set whose dimension is greater than unity projects into a set of positive Lebesgue measure in almost all directions.*

Theorem II *Any s-set whose dimension does not exceed unity projects into a set of dimension s in almost all directions.*
[dimension = Hausdorff; s-set = Borel set with $0 < \mathcal{H}^s(E) < \infty$]

John Marstrand's 1954 paper

• The paper notes that, because of a result of Davies (1952) the theorems can be expressed in terms of dimension:



Theorem I *Any set whose dimension is greater than unity projects into a set of positive Lebesgue measure in almost all directions.*

Theorem II *Any set of dimension s that does not exceed unity projects into a set of Hausdorff dimension s in almost all directions.*

• It follows easily from the definitions that for $E \subset \mathbb{R}^n$ $\dim_H \mathrm{proj}_\theta E \leq \min\{\dim_H E, 1\}$ for all θ. The almost sure lower bound requires more work.

• The paper was the first time the interplay between the geometry and dimensional properties of general 'fractals' were investigated – 20 years before the word 'fractal' was coined. 'Fractal Geometry' is now a flourishing research area.

• However, the paper attracted little attention for 30 years, except:

- The natural analogues for projections of sets in $\mathbb{R}^n$ to m-dimensional subspaces were proved by Mattila (1975).
- Kaufman (1968) gave a new proof of the Theorems using potential theory.

• In the last 30 years the paper has had an vast number of citations, with many variants, generalisations and specialisations

John Marstrand's 1954 paper - content

• Projection theorems

$+$ much more ...

• Intersection with lines – almost every line through almost every point of an s-set E $(s > 1)$ intersects E in a set of dimension $s - 1$.

• Projection of sets from points

• Examples to show results are best possible

• The density $\lim_{r\to 0} \mathcal{H}^s(E \cap B(x,r))/(2r)^s$ of an s-set $E \subset \mathbb{R}^2$ can only exist and equal 1 on a set of positive $\mathcal{H}^s$-measure if $s = 0, 1$ or 2

• Bounds on angular densities (i.e. densities in a sector)

• Discussion of weak tangents to sets

Energy characterisation of Hausdorff dimension

Marstrand's proof was geometric and quite intricate. Kaufman's (1968) potential theoretic proof has become the standard approach for such problems.

This depends on the following energy characterisation of Hausdorff dimension:

$$\dim_H E = \sup\Big\{ s : E \text{ supports a positive finite measure } \mu \text{ such that } \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^s} < \infty \Big\}.$$

Kaufman's proof of Marstrand's projection theorem

Suppose $\dim_H E > s$ where $s < 1$. Take a measure μ on E such that $\displaystyle\int\int \frac{d\mu(x)d\mu(y)}{|x-y|^s} < \infty$.

Write μ_θ for the projection of μ onto the line in direction θ, so $\int_{-\infty}^{\infty} f(t)d\mu_\theta(t) = \int_E f(x\cdot\theta)d\mu(x)$ for continuous f. Then

$$\begin{aligned}
\int_0^\pi \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\mu_\theta(t)d\mu_\theta(u)}{|t-u|^s}\right]d\theta &= \int_0^\pi \left[\int_E\int_E \frac{d\mu(x)d\mu(y)}{|x\cdot\theta - y\cdot\theta|^s}\right]d\theta \\
&= \int_E\int_E\int_0^\pi \frac{d\theta}{|\mathbf{u}_{x-y}\cdot\theta|^s}\frac{d\mu(x)d\mu(y)}{|x-y|^s} \\
&\leq c\int_E\int_E \frac{d\mu(x)d\mu(y)}{|x-y|^s} < \infty
\end{aligned}$$

Hence for almost all θ, $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\mu_\theta(t)d\mu_\theta(u)}{|t-u|^s} < \infty$, so, since μ_θ is supported by $\mathrm{proj}_\theta E$, we conclude $\dim_H \mathrm{proj}_\theta E \geq s$. $\square$

What about when $\dim_H E = 1$?

Marstrand's theorem does not say anything about the Lebesgue measure of projections of E when $\dim_H E = 1$.

However, Besicovitch (1928, 1938, 1939) had already considered projections of sets of positive finite 1-dimensional Hausdorff measure in the plane:

- If $E \subseteq \mathbb{R}^2$ and $0 < \mathcal{H}^1(E) < \infty$, then E may be decomposed into a regular part E_R and an irregular part E_I, so $E = E_R \cup E_I$ where

  E_R is a subset of a countable number of rectifiable curves
  E_I intersects every rectifiable curve in length 0.

- Then

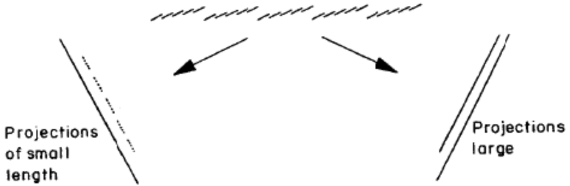
  $\mathcal{L}(\mathrm{proj}_\theta E_R) > 0$ for all except at most one value of θ
  (provided $\mathcal{H}^1(E_R) > 0$)
  $\mathcal{L}(\mathrm{proj}_\theta E_I) = 0$ for almost all θ.

What if $\dim_H E = 1$ and $\mathcal{H}^1(E) = \infty$?

Almost anything can happen!

Theorem *Given a set E_θ for each $0 \le \theta < \pi$ (+ measurability condition), there exists a Borel set $E \subset \mathbb{R}^2$ such that $\mathcal{L}(E_\theta \triangle \mathrm{proj}_\theta E) = 0$ for almost all directions θ.*

This follows by dualising a result of Davies (1952) on covering sets by lines. Alternatively, there is a direct 'iterated venetian blind' construction.



Higher dimensional analogues are also valid.

Digital sundial theorem (F, 1986)

Given a subset E_V of each 2-dimensional subspace V of $\mathbb{R}^3$ (+ measurability condition), there exists a Borel set $E \subset \mathbb{R}^3$ such that for almost all subspaces V $\mathcal{L}^2(E_V \triangle \mathrm{proj}_V E) = 0$.

Packing measure and dimension

Packing dimension was introduced by Tricot & Taylor (1982) as a 'dual' to Hausdorff dimension.

For $s > 0, \delta > 0, E \subseteq \mathbb{R}^n$, let

$$\mathcal{P}^s_\delta(E) = \sup\Big\{\sum_{i=1}^{\infty}(\operatorname{diam} B_i)^s : B_i \text{ disjoint balls with centres in } E, \operatorname{diam} B_i \leq \delta\Big\}.$$

Then

$$\mathcal{P}^s_0(E) = \lim_{\delta\to 0}\mathcal{P}^s_\delta(E)$$

and

$$\mathcal{P}^s(E) = \inf\Big\{\sum_{i=1}^{\infty}\mathcal{P}^s_0(E_i) : E \subseteq \bigcup_{i=1}^{\infty} E_i\Big\}$$

is the s-dimensional packing measure of E.

The packing dimension of $E \subseteq \mathbb{R}^n$ is

$$\dim_P E = \inf\{s : \mathcal{P}^s(E) = 0\} = \sup\{s : \mathcal{P}^s(E) = \infty\}.$$

For all $E \subseteq \mathbb{R}^n$

$$\dim_H E \leq \dim_P E.$$

Projections and packing dimension

• Projection theorems for packing dimension are more subtle than for Hausdorff dimension. In particular packing dimension is not a.s. preserved under projections:

• For $E \subseteq \mathbb{R}^n$, for projection onto almost all m-dimensional subspaces V,

$$\frac{\dim_P E}{1 + (1/m - 1/n)\dim_P E} \leq \dim_P \mathrm{proj}_V E \leq \min\{\dim_P E, m\}$$

(Järvenpää, 1994; Howroyd & F 1996)

• Is $\dim_P \mathrm{proj}_V E$ constant for almost all V?

• Yes! – Given $E \subseteq \mathbb{R}^n$, $\dim_P \mathrm{proj}_V E = \dim_P^m E$ for almost all m-dimensional subspaces V, where $\dim_P^s E$ is the packing dimension profile of E, reflecting how E typically appears when examined from an s-dimensional viewpoint. (Howroyd & F 1997)

Generalized projections and transversality

• The projection theorems may be developed to a much more general setting.

• Consider a family of maps $\pi_\theta : \mathbb{R}^n \to \mathbb{R}^m$ for θ in a suitable parameter space. If $\int \frac{d\theta}{|\pi_\theta(x)-\pi_\theta(y)|^s} \leq \frac{c}{|x-y|^s}$ then Kaufman's argument goes through to give $\dim_H \pi_\theta E = \min\{m, \dim_H E\}$ for almost all θ.

• The family is π_θ is transversal if $|\pi_\theta(x) - \pi_\theta(y)|/|x-y|$ and $\frac{\partial}{\partial\theta}|\pi_\theta(x) - \pi_\theta(y)|/|x-y|$ are not simultaneously small.

Peres & Schlag (2000) showed that for a transversal family

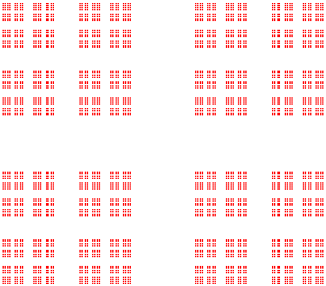
$$\dim_H \pi_\theta(E) = \min\{m, \dim_H E\} \text{ for almost all } \theta.$$

• Method generalises the potential-theoretic and Fourier methods beyond recognition, to get information on exceptional sets, function spaces, etc.

• Many applications, including Bernoulli convolutions, sums of Cantor sets, pinned distance sets, etc.

Size of exceptional sets

Marstrand's theorem tells nothing about which particular directions θ have projections with dimension smaller than normal, i.e. when $\dim_H \text{proj}_\theta E < \min\{\dim_H E, 1\}$.



The set shown has dimension $\log 4/\log 5/2 = 1.51$, but with some projections of dimension < 1.

Size of exceptional sets

However, the set of exceptional directions can't be too big:

- (Kaufman, 1968) If $E \subseteq \mathbb{R}^2$ and $\dim_H E \leq 1$,

$$\dim_H\{\theta : \dim_H \mathrm{proj}_\theta E < \dim_H E\} \leq \dim_H E.$$

– Follows by adapting Kaufman's proof above, since if $s < \dim_H E < t$ and T is any set of θ of dimension t, we may find a measure ν supported by T such that $\int_T |\mathbf{u}_{x-y} \cdot \theta|^{-s} d\nu(\theta) \leq M < \infty$.
Then $\dim_H \mathrm{proj}_\theta E \geq s$ for ν almost all $\theta \in T$.

- (F, 1982) If $E \subseteq \mathbb{R}^2$ and $\dim_H E > 1$,

$$\dim_H\{\theta : \mathcal{L}(\mathrm{proj}_\theta E) = 0\} \leq 2 - \dim_H E.$$

– all known proofs involve Fourier transforms.

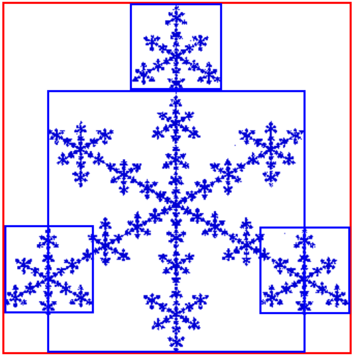
Projections of specific sets

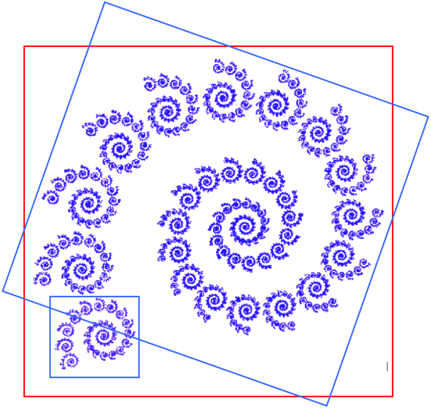
Recently projections and the exceptional set of directions have been investigated for *specific* sets where it is sometimes possible to identify the exceptional directions

– or perhaps to show that there aren't any exceptional directions at all.

Self-similar sets

A family of contracting similarities $\{S_1, \dots, S_m\}$ on $\mathbb{R}^n$ defines a unique non-empty compact set E such that $E = \cup_{i=1}^m S_i(E)$. The set E is called self-similar.





If S_i has similarity ratio r_i then $\dim_H E = s$ where $\sum_{i=1}^m r_i^s = 1$, provided there is 'not too much overlapping'.

Projections of specific self-similar sets

The behaviour of the dimensions $\dim_H \mathrm{proj}_\theta E$ as θ varies has been investigated in certain specific cases. For example:

Let E be the 1-dimensional Sierpínski triangle, so $\dim_H E = 1$.

For projections in direction θ:

(a) if $\theta = p/q$ is rational,

and $p + q \not\equiv 0 \pmod 3$

$\dim_H \mathrm{proj}_\theta E < 1$;

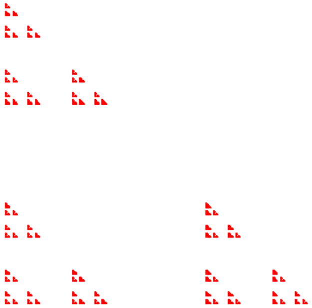
and $p + q \equiv 0 \pmod 3$

$\mathrm{proj}_\theta E$ contains an interval,

(b) if θ is irrational,

$\mathcal{L}(\mathrm{proj}_\theta E) = 0$.

(Kenyon, 1997)



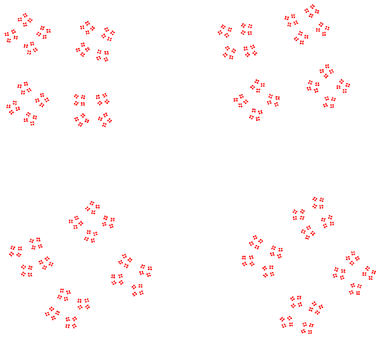
Similar investigations have been done for certain other 'regular' sets.

Self-similar sets with rotations

Now suppose the similarities on $\mathbb{R}^n$ may be written

$$S_i(\mathbf{x}) = r_i O_i(\mathbf{x}) + \mathbf{t}_i$$

where $0 < r_i < 1$ is the scale factor, O_i is a rotation and $\mathbf{t}_i$ is a translation. We say that the family $\{S_1, \ldots, S_m\}$ has dense rotations if the group generated by $\{O_1, \ldots, O_m\}$ is dense in the $SO(n, \mathbb{R})$.



Theorem (Peres & Shmerkin, 2008, Hochman & Shmerkin, 2012)
Let $E \subset \mathbb{R}^n$ be a self-similar set defined by a family $\{S_1, \ldots, S_m\}$ of similarities with dense rotations. Then
$\dim_H \mathrm{proj}_V E = \min\{\dim_H E, m\}$ for **all** m-dimensional subspaces V.

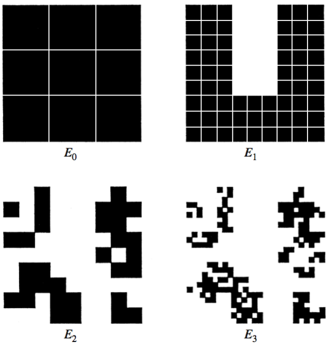
Self-similar sets with rotations

Theorem (Peres & Shmerkin, 2008, Hochman & Shmerkin, 2012) Let $E \subset \mathbb{R}^n$ be a self-similar set defined by a family $\{S_1, \ldots, S_m\}$ of similarities with dense rotations. Then $\dim_H \mathrm{proj}_V E = \min\{\dim_H E, m\}$ for **all** m-dimensional subspaces V.

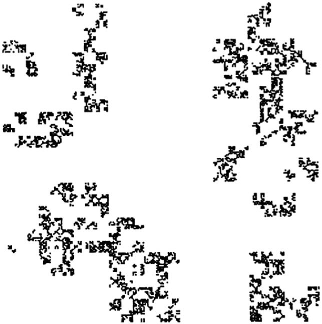
The proof of this involves setting up a CP-chain ('conditional probability'), that is a measure-valued Markov process representing the renormalised measures scaled up about points of the set E. The chain is ergodic, on a space which includes Haar measure on $SO(n, \mathbb{R})$ as a factor. Projecting the measure onto subspaces gives an ergodic sequence of measures on each subspace. An entropy-like expression which is continuous with respect to subspaces and approximates Hausdorff dimension from below gives the continuity of the dimension of projection, and so extends 'almost all θ' to 'all θ' in Marstrand's Theorem.

Fractal percolation

Squares are repeatedly divided into $M \times M$ subsquares and each square is retained independently with probability p.



Percolation process



The percolation set

Conditional on non-extinction this yields a percolation set E_p of dimension a.s. $\log 9p/\log 3$ ($= 1.54$ if $p = 0.6$ as in the picture).

Fractal percolation

Let $E_p \subseteq \mathbb{R}^2$ be the random percolation set based on an $M \times M$ subdivision with each square retained with probability p.

• If $p > 1/M^2$ then there is a positive probability of $E_p \neq \emptyset$, in which case $\dim_H E_p = \log M^2 p / \log M$ almost surely.

• It is immediate from Marstrand's theorems that, almost surely conditional on $E_p \neq \emptyset$, for almost all θ,

$$\dim_H \mathrm{proj}_\theta E_p = \min\{\dim_H E_p, 1\};$$

$$\text{if } \log M^2 p / \log M > 1 \text{ then } \mathcal{L}(\mathrm{proj}_\theta E_p) > 0.$$

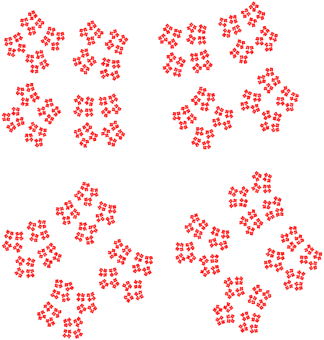
Theorem (Rams & Simon, 2013, 2014) Almost surely, conditional on $E_p \neq \emptyset$, for **all** θ

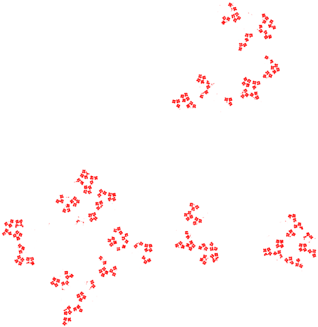
$$\dim_H \mathrm{proj}_\theta E_p = \min\{\dim_H E_p, 1\};$$

$$\text{if } \log M^2 p / \log M > 1 \text{ then } \mathrm{proj}_\theta E_p \text{ contains an interval.}$$

Fractal percolation on a self-similar set with rotations

Now let $E \subset \mathbb{R}^n$ be a self-similar set defined by a family $\{S_1, \ldots, S_m\}$ of similarities with dense rotations. E has a natural hierarchical construction.





Perform the percolation process on E with respect to its hierarchical construction, with each similar component retained independently with probability p, to give a percolation set $E_p \subseteq E$.

Fractal percolation on a self-similar set with rotations

Theorem (Jin & F, 2014) Conditional on $E_p \neq \emptyset$, almost surely, for **all** θ

$$\dim_H \mathrm{proj}_\theta E_p = \min\{\dim_H E_p, 1\}.$$

The proof uses ergodic theory on the space $(\Lambda^{\mathbb{N}}, \Omega^*, SO(n,\mathbb{R}))$. Here $\Lambda = \{1, 2, \ldots, m\}$ and Ω^* is a product of copies of the probability space underlying the percolation process, indexed by $\cup_{k=1}^{\infty}\Lambda^k$. The map

$$T : (\mathbf{i}, \omega_{\mathbf{j}}, g) \mapsto (\sigma(\mathbf{i}), \omega_{\mathbf{i}|1\mathbf{j}}, gO_{\mathbf{i}|1})$$

is invariant, ergodic and mixing with respect to an appropriate measure (Peyrière $\times$ Haar). Then, with $M : (\mathbf{i}, \omega_{\mathbf{j}}, g) \mapsto g\nu$, we get $\{M \circ T^n\}_n$ a stationary ergodic sequence.

Then CP-trees are used again to get almost sure continuity of the projected measures.

Other recent extensions and variants

• Projections onto restricted families of subspaces (Orponen, Fässler, Järvenpää, Keleti, Leikas, ...)

• Projections in infinite dimensional spaces (Hunt & Kaloshin)

• Projections in Heisenberg groups (Mattila, Balogh, Tyson, ...)

• Applications to sums and products (projections of $E \times F$ correspond to sums $E + \lambda F$) (Moreira, Lima, Peres, Shmerkin, ...)

• Equal dimensions of projections in all directions (apart from obvious exceptions) for products of Cantor sets, and of self-affine carpets (Ferguson, Jordan, Shmerkin, Peres)

• Natural projection from its tangent bundle to a Riemann surface (Ledrapier, Lindenstrauss, Järvenpää, Leikas ,...)

• Multifractal projection results (Olsen, Barral, Bhouri, ...)

• ......