Lowest fractal dimensions for universal differentiability

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Universal Differentiability Set (UDS)

A Borel set $S \subseteq X$ is a UDS if for every Lipschitz function $f : X \to \mathbb{R}$ there is an $x \in S$ such that f is (Fréchet) differentiable at x.

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Sharpness of the result, $n \geq 2$

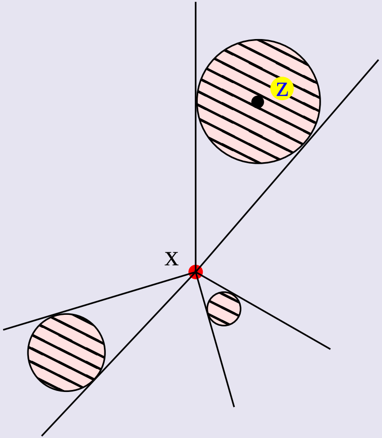
[Preiss, 1990] [Alberti, Csörnyei, Preiss 2010] [Doré–M., 2010, '11, '12] [Dymond–M., 2013] [Preiss–Speight, 2013] [Csörnyei, Jones 2013]
If $n \geq 2$, then $\mathbb{R}^n$ contains Lebesgue null universal differentiability subsets.

Examples of non-universal differentiability sets

Classical results

1. $E \subseteq X$ is porous.

Def. Let $\lambda > 0$. $E \subseteq X$ is λ-porous at $x \in X$ if for every $r > 0$ there is a $z \in B(x, r)$ such that $B(z, \lambda\|z - x\|) \cap E = \emptyset$.



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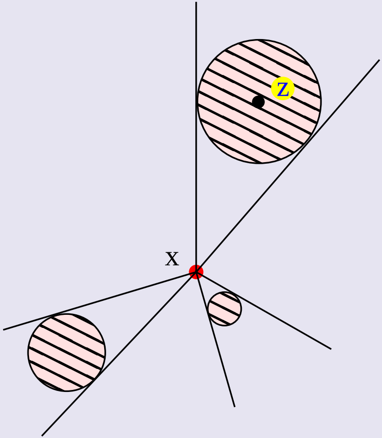

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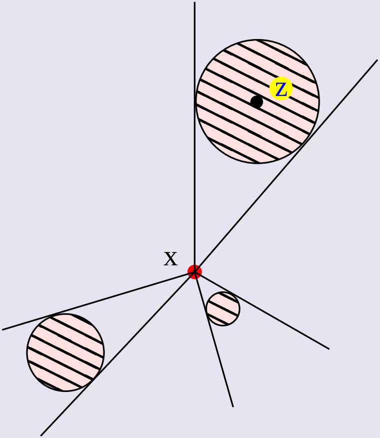

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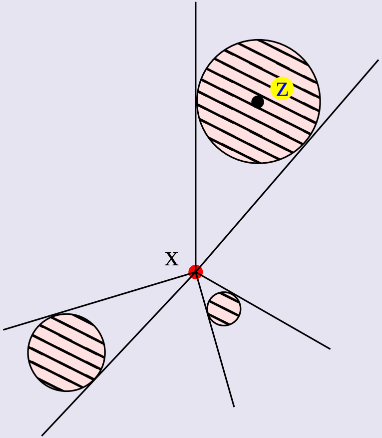

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$E \subseteq X$ is porous if $\exists\lambda > 0$ s.t. it is λ-porous at each of its points.

Further examples of non-UDS and UDS

Classical results: non-UDS

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Hausdorff and Minkowski dimension

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Hausdorff dimension

$$\mathcal{H}^p(A) = \lim_{\delta \downarrow 0} \inf \left\{ \sum_i \mathrm{diam}(E_i)^p : A \subseteq \bigcup_i E_i, \mathrm{diam}(E_i) \leq \delta \right\}.$$

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Minkowski (box counting) dimension

Now for each $\delta > 0$ let N_δ be the minimal possible number of balls of radius δ with which it is possible to cover A. Then

$$\underline{\overline{\dim}_{\mathcal{M}}(A)/\underline{\dim}_{\mathcal{M}}(A) = \inf\{p : \overline{\lim}_{\delta\downarrow 0}/\underline{\lim}_{\delta\downarrow 0} N_\delta \delta^p = 0\}}$$

is the upper (lower) Minkowski dimension of A.

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Approximation property (Dymond–O.M. 2013; Dymond 2014)

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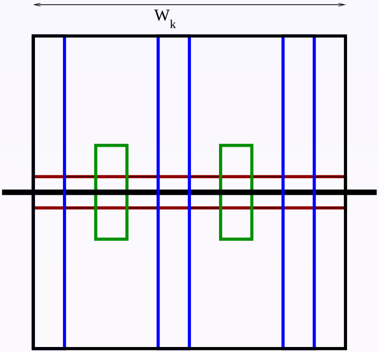
Weak Conjecture

E UDS, $\varepsilon > 0$, $x \in \ker(E) \implies \exists \gamma$, $\|\gamma' - e\| < \varepsilon$ with $|\gamma^{-1}(E)| > 0$

Construction



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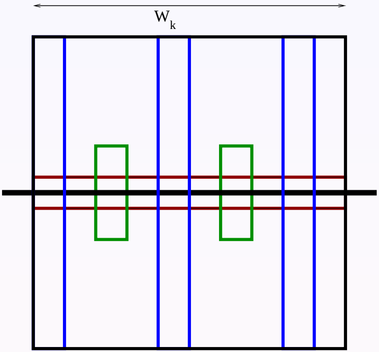

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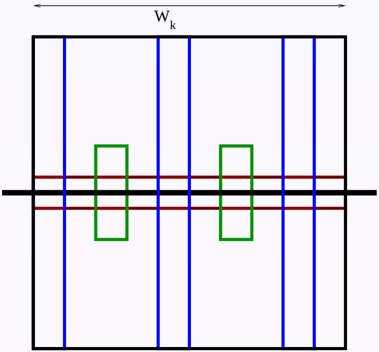

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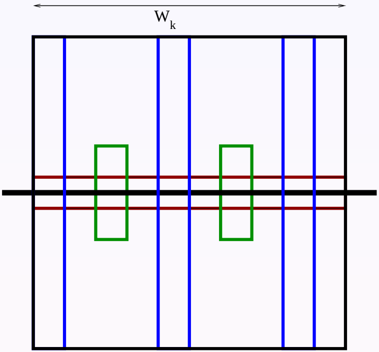

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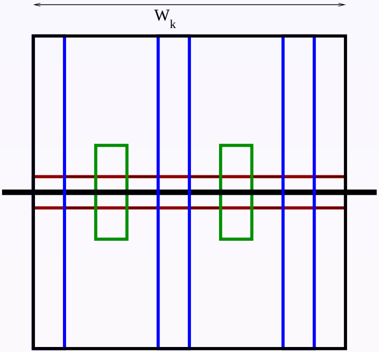

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As $p > 1$,

$$\frac{N_{w_{k+1}} w_{k+1}^p}{N_{w_k} w_k^p} \leq (\log R)^{2m+1} R^{1-p} < 1,\ R \text{ large}$$

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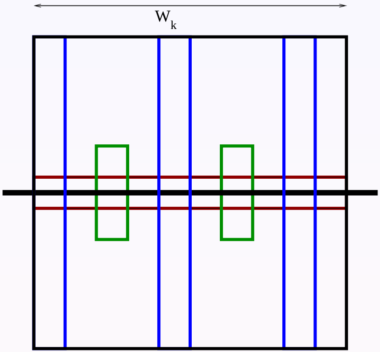

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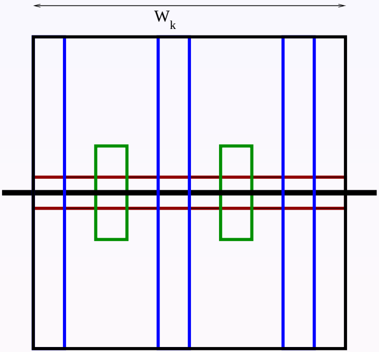

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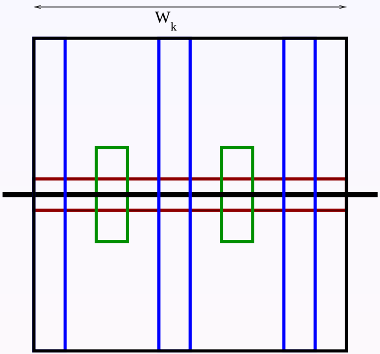
We show: $N_{w_{k+1}} w_{k+1}^p R_{k+1}^p \to 0$

Further ideas



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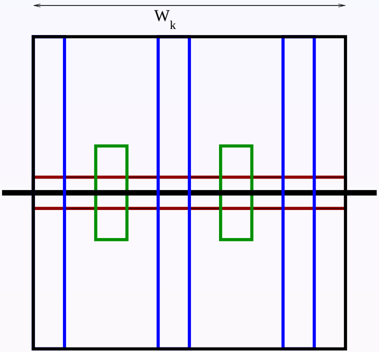

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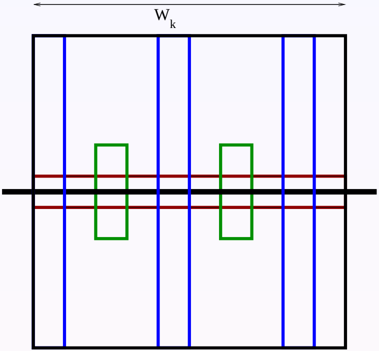

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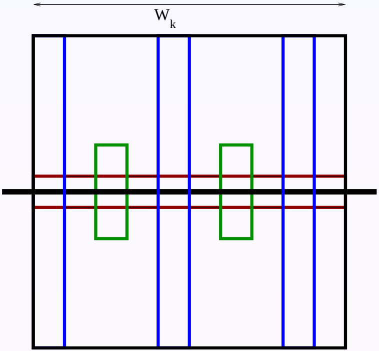

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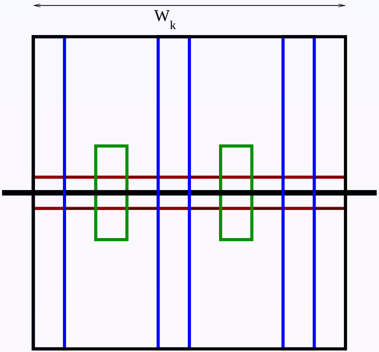
Conjecture

Let $\mathcal{F} = \{f(x) = o(x), x \to 0\}$.
If N is a UDS and $\mathcal{F}(N) = \{f \in \mathcal{F} : \mathcal{M}_f(N) < \infty\}$ then
$\exists f_0 \in \mathcal{F}$ s.t. $f = o(f_0)\ \forall f \in \mathcal{F}(N)$.

Further ideas



Can we get $N_{w_{k+1}} \leq N_{w_k} \times R\Phi(R)$ for *any* $\Phi(R) \nearrow \infty$ chosen in advance?

Describe the class of gauge functions f for which
$\overline{\mathcal{M}_f}(N) = \overline{\lim}_{\delta>0} N_\delta f(\delta)$ or
$\underline{\mathcal{M}_f}(N) = \underline{\lim}_{\delta>0} N_\delta f(\delta)$
is finite.

We know $\lim_{\delta\to 0} N_\delta \delta$ is infinite.

Conjecture

Let $\mathcal{F} = \{f(x) = o(x), x \to 0\}$.
If N is a UDS and $\mathcal{F}(N) = \{f \in \mathcal{F} : \mathcal{M}_f(N) < \infty\}$ then
$\exists f_0 \in \mathcal{F}$ s.t. $f = o(f_0)\ \forall f \in \mathcal{F}(N)$.

If $f_0 \in \mathcal{F}$ then there exists a UDS N such that $f = o(f_0)\ \forall f \in \mathcal{F}(N)$.

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Conjectures

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2. Every UDS contains a closed universal differentiability subset.