Restricted families of projections

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Restricted families of projections?

Notation for the talk:

- V stands for a k-dimensional subspace of $\mathbb{R}^d$.
- The collection of such V's is denoted by $\mathcal{G}(d, k)$.
- π_V is the orthogonal projection $\pi_V \colon \mathbb{R}^d \to V$.

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Definition (Restricted families of projections)

Take a strict subset $\mathcal{G} \subsetneq \mathcal{G}(d,k)$. The family of projections $(\pi_V)_{V \in \mathcal{G}}$ is a *restricted family of projections*.

Hopes vs. reality

What one hopes to prove:

- **Suitable** restricted families of projections admit a Marstrand-type theorem: $\dim \pi_V(K) = \min\{\dim K, k\}$ for some – or "almost all" – $V \in \mathcal{G}$.

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What one can prove at the moment (for certain families in $\mathbb{R}^3$):

- The above holds, if $\dim K$ is small enough, typically much smaller than k (easy). If $\dim K$ is **not** small enough, there's an ε-improvement over the easy bound.

The easy part follows by classical methods. The ε-improvement is achieved by looking at the structure of (hypothetical) extremizers.

The classical argument

What follows is a low-detail review of the classical argument, which gives the projection theorem for full families of projections. For completeness, here's the result:

Theorem (Marstrand-Mattila projection theorem, 1954,1975)

$\dim \pi_V(K) = \min\{\dim K, k\}$ *for a.e.* $V \in \mathcal{G}(d,k)$*.*

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- For purposes of illustration, I will focus on a discrete variant:

Theorem

"If K *is a set of* δ*-separated points satisfying a non-concentration condition, then there are many subspaces* V *such that* $\pi_V(K)$ *contains almost* card K *points."*

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Theorem

"If K is a set of δ-separated points satisfying a non-concentration condition, then there are many subspaces V such that $\pi_V(K)$ contains almost card K *points."*

- The required non-concentration is the following:

$$\sum_{x \neq y} \left(\frac{\delta}{|x-y|} \right)^{\dim K} \ll \delta^{-\dim K}$$

The classical argument II

- If V is s.t. $|\pi_V(x) - \pi_V(y)| \geq \delta$ for all $x \neq y$ in K, then $\pi_V(K)$ obviously contains $(\operatorname{card} K)$ δ-separated points.

The classical argument II

- If V is s.t. $|\pi_V(x) - \pi_V(y)| \geq \delta$ for all $x \neq y$ in K, then $\pi_V(K)$ obviously contains (card K) δ-separated points.
- So, the enemy is the event

$$E(x,y) := \{V \in \mathcal{G}(d,k) : |\pi_V(x) - \pi_V(y)| < \delta\}.$$

- The key of the whole proof is that this is rare event. If $\gamma_{d,k}$ is the natural measure on $\mathcal{G}(d,k)$, then

$$\gamma_{d,k}(E(x,y)) \lesssim \left(\frac{\delta}{|x-y|}\right)^k.$$

The classical argument III

The proof is now completed by double-counting:

- If $\operatorname{card} \pi_V(K) \ll \operatorname{card} K$, then $|\pi_V(x) - \pi_V(y)| < \delta$ for many pairs $x \neq y$. In other words,

$$\sum_{x \neq y} \chi_{E(x,y)}(V) \quad \text{is big.}$$

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- This cannot happen for too many V's, because

$$\mathbb{E}_V \sum_{x \neq y} \chi_{E(x,y)}(V) = \sum_{x \neq y} \gamma_{d,k}(E(x,y)) \lesssim \sum_{x \neq y} \left(\frac{\delta}{|x-y|} \right)^{\dim K},$$

using first $\gamma_{d,k}(E(x,y)) \lesssim (\delta/|x-y|)^k$ and then $\dim K \leq k$.

An abstraction

- For later use, let's record the following abstract projection theorem, which follows from the previous proof.
- Let (Λ, γ) be probability space, and let $(\pi_\lambda)_\lambda \colon \mathbb{R}^d \to \mathbb{R}^m$ be a collection of 1-Lipschitz linear mappings satisfying

$$\gamma(\{\lambda : |\pi_\lambda(x)| < \delta\}) \lesssim (\delta/|x|)^r.$$

Theorem (Abstract projection theorem (APT))

$\dim \pi_\lambda(K) = \min\{\dim K, r\}$ *for* γ*-a.e.* λ.

The problem with restricted families

- The preceding projection theorems relied on uniform sub-level set estimates $\gamma(\{\lambda : |\pi_\lambda(x)| < \delta\}) \lesssim (\delta/|x|)^r$.
- In the restricted situation one also has sub-level set estimates, but the constants and the sharp rates of decay depend on the position of the point x:

$$\gamma(\{\lambda : |\pi_\lambda(x)| < \delta\}) \leq C(x) \cdot (\delta/|x|)^{r(x)}.$$

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Restricted families with sharp APT

- First, consider the subfamily $\mathcal{G} \subset G(3,1)$ of all lines contained in the *xy*-plane. What is the best possible (uniform) sub-level set estimate?

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$$\{L \in \mathcal{G} : \pi_L(x) = 0\} = \mathcal{G}.$$

- So, there's no possibility of uniform decay like

$$\gamma(\{L \in \mathcal{G} : |\pi_L(x)| \leq \delta\}) \lesssim (\delta/|x|)^r, \quad r > 0, \ x \in \mathbb{R}^3,$$

and the APT gives nothing useful. There's also nothing to be had: the 1-dimensional set $K = \{z\text{-axis}\}$ π_L-projects to $\{0\}$ for all $L \in \mathcal{G}$.

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- HOWEVER: for $x \notin \{z - \text{axis}\}$ one has

$$\gamma(\{L \in \mathcal{G} : |\pi_L(x)| \leq \delta\}) \lesssim_x \delta/|x|.$$

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$$\gamma_{\mathcal{G}}(\{V \in \mathcal{G} : |\pi_V(x)| \leq \delta\}) \lesssim \delta/|x|.$$

  The uniformly distributed measure achieves that.
- Hence, the APT promises dimension conservation for up to 1-dimensional sets. Again, that's the best you can get, because any subset K of the xy-plane projects inside the line $V \cap \{xy - \text{plane}\}$ for all $V \in \mathcal{G}$.

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- HOWEVER: for $x \notin \{xy - \text{plane}\}(\delta)$ one has

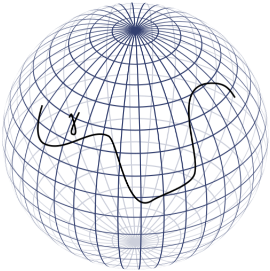
$$\{V \in \mathcal{G} : |\pi_V(x)| \leq \delta\} = \emptyset!.$$

Restricted families of projections

- The APT is sharp in the preceding examples, because there are certain subspaces (z-axis and xy-plane) in $\mathbb{R}^3$, where the sub-level set estimates are uniformly poor.
- We want to get rid of this phenomenon, so we add some curvature. Consider a smooth curve $\eta \colon (0,1) \to S^2$, satisfying

$$\operatorname{span}\{\eta(t), \dot{\eta}(t), \ddot{\eta}(t)\} = \mathbb{R}^3, \qquad t \in (0,1).$$

- Something like this:



Restricted families of projections

- Then, we get a family of lines and planes by setting

$$\mathcal{G}_L(\eta) := \{\text{span}\{\eta(t)\} : t \in (0,1)\}$$

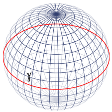
and

$$\mathcal{G}_V(\eta) := \{\text{span}\{\eta(t)\}^{\perp} : t \in (0,1)\}.$$

- The examples above were $\mathcal{G}_L(\eta)$ and $\mathcal{G}_V(\eta)$, corresponding to the curve η parametrising the unit circle on the *xy*-plane. But of course

$$\text{span}\{\eta(t), \dot{\eta}(t), \ddot{\eta}(t)\} = xy - \text{plane}$$

for $t \in (0,1)$, so the curvature requirement excludes these examples.



Restricted families of projections

- Assuming the curvature condition, counterexamples become very evasive: the regions of poor sub-level set estimates are no longer subspaces. The following conjecture seems plausible:

Conjecture

The projections π_L, $L \in \mathcal{G}_L(\eta)$, should conserve dimension for up to 1-dimensional sets, and the projections π_V, $V \in \mathcal{G}_V(\eta)$ should conserve dimension for up to 2-dimensional sets.

Kaufman's method

- Sanity check: does it follows from the APT?
- The projection families $\mathcal{G}_L(\eta)$ and $\mathcal{G}_V(\eta)$ are parametrised by $(0,1)$, so the natural choice for $\gamma_{\mathcal{G}}$ on both manifolds is essentially $\mathcal{L}^1|_{(0,1)}$. The bounds below are the worst-case sub-level set estimates (i.e. they are valid for all and sharp for certain $x \in \mathbb{R}^3$):

$$\mathcal{L}^1(\{L \in \mathcal{G}_L(\eta) : |\pi_L(x)| \leq \delta\}) \lesssim (\delta/|x|)^{1/2},$$

and

$$\mathcal{L}^1(\{V \in \mathcal{G}_V(\eta) : |\pi_V(x)| \leq \delta\}) \lesssim \delta/|x|.$$

Kaufman's method

- The bounds have the following corollary:

Corollary (to the APT)

The projections π_L, $L \in \mathcal{G}_L(\eta)$, conserve dimension for up to $1/2$-dimensional sets, and the projections π_V, $V \in \mathcal{G}_V(\eta)$, conserve dimension for up to 1-dimensional sets.

- It's worth observing that the "$1/2$" already improves on the non-curved case (where no positive result was to be had), but the "1" doesn't.

Small improvements

- Nevertheless, the "1" is not the end of the story here:

Theorem (Fässler, O. (2013))

For every $s > 1$, there is $\sigma(s) > 1$ such that the following holds. If $\dim K = s$, then $\dim \pi_V(K) \geq \sigma(s)$ for almost every $V \in \mathcal{G}_V(\eta)$.

- For $\mathcal{G}_L$, we obtained the same result for the *packing dimension* of projections, but the Hausdorff dimension narrowly escaped. Except for this special curve:

$$\eta(t) = (\cos(t), \sin(t), 1).$$

Theorem (O. (2013))

For this special curve η, the previous theorem holds with $\mathcal{G}_V$ replaced by $\mathcal{G}_L$ and "1" replaced by "$1/2$".

The proof in four slides (1)

- Recall: we're interested in the one-dimensional family of 2-dim subspaces given by

$$V_t := \text{span}\{\eta(t)\}^{\perp}, \qquad t \in (0,1).$$

- As earlier, let K be a finite δ-separated set containing $\sim \delta^{-s}$ points, $s > 1$, satisfying an s-dimensional non-concentration property.
- The goal is to find many V_t's such that $\pi_{V_t}(K)$ contains $\gg \delta^{-1}$ δ-separated points.
- The APT strategy boils down to estimating

$$\sum_{x \neq y} |\{t : |\pi_{V_t}(x-y)| \leq \delta\}|$$

- As we already know, the best general bound is

$$|\{t : |\pi_{V_t}(x-y)| \leq \delta\}| \lesssim \frac{\delta}{|x-y|}.$$

The proof in four slides (2)

- The plan is to exploit the fact that this bound can be improved a lot, unless $(x - y)$ has a rather special orientation, namely $(x - y) \in (V_t^{\perp})(\delta)$ **for some t.**
- If $(x - y) \notin (V_t^{\perp})(\delta)$ for any t, then indeed

$$\{t : |\pi_{V_t}(x - y)| \leq \delta\} = \emptyset!$$

- This leads us to consider a "counter-assumption": suppose that the sum

$$\sum_{x \neq y} |\{t : |\pi_{V_t}(x - y)| \leq \delta\}|$$

is roughly as large as the "general $x - y$" estimate allows. Can we describe the structure of K?

The proof in four slides (3)

- Quite easily, in fact, and here's the answer:
- If the sum is almost as large as it can be (in view of the "general bound"), then there's a δ^{κ}-proportion of the points x such that a δ^{κ}-proportion of the set K is contained in a δ-neighbourhood of

$$x + C := x + \bigcup_{t \in (0,1)} V_t^{\perp}.$$

- Here $\kappa \searrow 0$, as the counter-assumption gets stronger.
- C is a conical surface of some sort, and $C(\delta)$ will stand for its δ-neighbourhood.

The proof in four slides (4)

- Almost done: since a large part of K is contained in many $(x + C(\delta))$'s...
- ...a large part of K is actually contained in $(x + C(\delta)) \cap (y + C_j(\delta))$ for some $x \neq y$!
- We can also choose $x \neq y$ relatively far apart.
- How does $(x + C(\delta)) \cap (y + C(\delta))$ look like? Since $(x + C) \cap (y + C)$ is the intersection of two conical surfaces, it's something essentially one-dimensional.
- This is a bit tedious to prove, but the upshot is that we've managed to cram a large part of an s-dimensional discrete set K inside an essentially one-dimensional set. That's not possible, since $s > 1$.

Further results

- The "restricted families of projections" problem in $\mathbb{R}^3$ is closely related to Fourier restriction questions.
- D. and R. Oberlin wrote a paper about this last year:

Theorem (D. and R. Oberlin, 2013)

Assuming the curvature condition,

$$\dim \pi_{V_t}(K) \geq \frac{3 \dim K}{4}$$

for almost all $t \in (0,1)$. *If* $\dim K \geq 2$, *the lower bound can be improved to* $\min\{\dim K - 1/2, 2\}$.

Restricted families in $\mathbb{R}^2$?

Question

Is there a "restricted families of projections" phenomenon in $\mathbb{R}^2$? For instance does there exist a collection of lines $\mathcal{L} \subset \mathcal{G}(2,1)$ with the following properties:

(a) $\dim \mathcal{L} < 1$,

(b) for any compact $K \subset \mathbb{R}^2$ with $\dim K = 1$, there exists $L \in \mathcal{L}$ with $\dim \pi_L(K) = 1$.

Question

Same as above, but replace (a) by

(a') $\dim \mathcal{L} = 0$.

Restricted families in $\mathbb{R}^2$?

For this problem, even a purely discrete variant is wide open. There are many ways to formulate this, for example:

Question

Call a family of lines $\mathcal{L} \subset \mathcal{G}(2,1)$ n-good, if for any n-point set $P \subset \mathbb{R}^2$ there exists $L \in \mathcal{L}$ such that $\operatorname{card} \pi_L(P) \geq n^{3/4}$. How small sets $\mathcal{L}$ can be n-good?

Proposition

It follows from Szemerédi-Trotter that ***any*** *collection $\mathcal{L}$ with* $\operatorname{card} \mathcal{L} \gg n^{1/2}$ *is n-good. On the other hand, it follows from a construction of Elekes-Erdős that no collection $\mathcal{L}$ of fixed size $C \in \mathbb{N}$ is n-good for large n.*

Conjecture

A random collection of $\sim \log n$ *lines is n-good.*

Why is no collection of fixed size n-good?

- The following construction was pointed out to me by András Máthé.
- Given any finite set $K = \{k_1, \ldots, k_C\} \subset \mathbb{R}$, a construction of Elekes-Erdős says that there exists an n-point set $A \subset \mathbb{R}$ (for some large n) containing $\gtrapprox n^2$ homothetic copies of K.
- In other words, there are $\gtrapprox n^2$ pairs $(x, y) \in \mathbb{R}^2$ such that $x + yK \subset A$.
- Now, let P be the set of these pairs, and note that $\pi_{L_j}(P) \subset A$ for all $L_j := \text{span}\{(1, k_j)\}$, $1 \leq j \leq C$. In particular,

$$\text{card}\, \pi_{L_j}(P) \leq n = (n^2)^{1/2} \lessapprox (\text{card}\, P)^{1/2}, \qquad 1 \leq j \leq C.$$

Sharpening Kaufman's bound?

The following is an instance of the well-known exceptional set estimate due to R. Kaufman (1968):

Theorem

Let $K \subset \mathbb{R}^2$ *be a compact set with* $\dim K = 1$. *Then*

$$\dim\{L \in \mathcal{G}(2,1) : \dim \pi_L(K) \leq s\} \leq s, \quad 0 \leq s \leq 1.$$

By a result of Bourgain (2003, 2010), this is not sharp for $s \sim 1/2$. In fact,

$$\dim\{L : \dim \pi_L(K) \leq s\} \searrow 0, \quad \text{as } s \searrow 1/2.$$

Question

Is Kaufman's bound sharp for any $1/2 < s < 1$? It **is** sharp for $s = 1$ (Kaufman-Mattila 1975).

Sharpening Kaufman's bound?

In the discrete world, Szemerédi-Trotter gives a tight estimate:

Proposition

Assume $P \subset \mathbb{R}^2$ *with* card $P = n$. *Then*

$$\text{card}\{L : \text{card}\,\pi_L(P) \le n^s\} \lesssim n^{2s-1}, \qquad 1/2 \le s < 1.$$

In the light of this bound and Bourgain's result, it is reasonable to conjecture that

Conjecture

$$\dim\{L : \dim \pi_L(P) \le s\} \le 2s - 1, \quad 1/2 \le s \le 1.$$

Sharpening Kaufman's bound?

This is probably hopeless, but even improving on Kaufman's bound by an $\varepsilon = \varepsilon(s)$ would be very interesting. Even this appears quite hard, however, because it would imply the following result in continuous sum-product theory:

Conjecture

Let $A \subset \mathbb{R}$ be a $1/2$-dimensional compact set, and let B be an s-dimensional compact set with $1/2 < s < 1$. Then

$$\dim(A + BA) > s.$$

This follows from Bourgain's work for $s \sim 1/2$, but not for, say $s = 3/4$. More generally,

Conjecture

If $\dim A, \dim C > 0$ and $\dim B < 1$, then

$$\dim(A + BC) > \dim B.$$

Projections and multi-scale analysis?

It is rather difficult to construct sets K such that $\dim K = 1$, and $\overline{\dim}_B \pi_L(K) < 1$ for many $L \in \mathcal{G}(2, 1)$.

Conjecture

If $\dim K = 1$, *then*

$$\dim\{L : \overline{\dim}_B \pi_L(K) < 1\} = 0.$$

- I only know that the exceptional set above can be uncountable.
- More generally, how to exploit "multi-scale information" in projection problems – i.e. the assumption that $\pi_L(K)$ is small on many (and not too rare) scales simultaneously?

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