Projections of fractal percolations II

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5. Non-homogeneous Fractal percolation sets
6. Homogeneous percolation of small dimension
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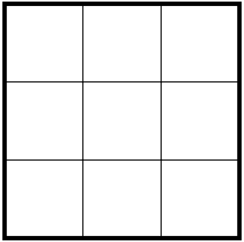
Fractal percolation, introduced by Mandelbrot early 1970's:

We partition the unit square into M^2 congruent sub squares each of them are independently retained with probability p and discarded with probability $1 - p$. In the squares retained after the previous step we repeat the same process at infinitum.



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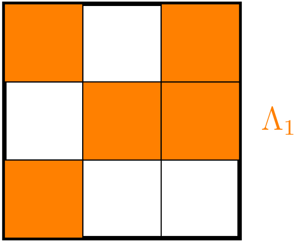

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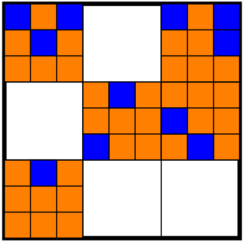

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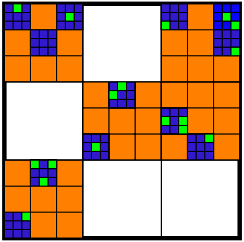

Λ_3

Let Λ_n be the union of the level n retained squares. Then the statistically self-similar set of interest is:

$$\Lambda := \bigcap_{n=1}^{\infty} \Lambda_n.$$

It was proved by Falconer and independently Mauldin, Willims that conditioned on non-extinction:

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \frac{\log(M^2 \cdot p)}{\log M} \text{ a.s.}$$

The expected number of descendants of every square is: $M^2 \cdot p$. Therefore, if $M^2 \cdot p < 1$ then $\Lambda = \emptyset$ a.s.

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So, we have almost surely:

- If $p \leq 1/M^2$ then $\Lambda = \emptyset$.
- If $1/M^2 < p < 1/M$ then $\dim_{\mathrm{H}}(\Lambda) < 1$ (but $\Lambda \neq \emptyset$ with positive probability).
- If $p > \frac{1}{M}$ then either
  - (a) $\Lambda = \emptyset$ or
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Theorem 1 (Marstrand)

Let $B \subset \mathbb{R}^2$ be a Borel set.

1. *If $\dim_{\mathrm{H}}(B) \leq 1$ then for $\mathcal{L}\mathrm{eb}$-a.e. θ, we have*

$$\dim_{\mathrm{H}}(\mathrm{proj}_{\theta}(B)) = \dim_{\mathrm{H}}(B)$$

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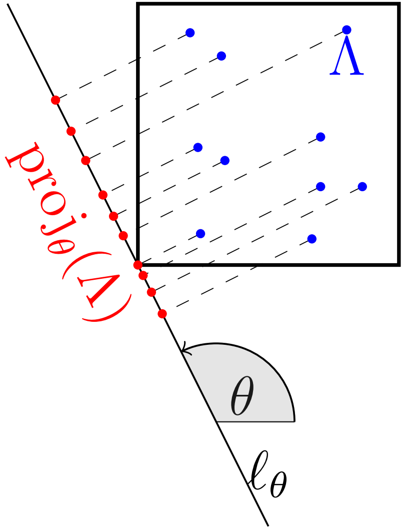
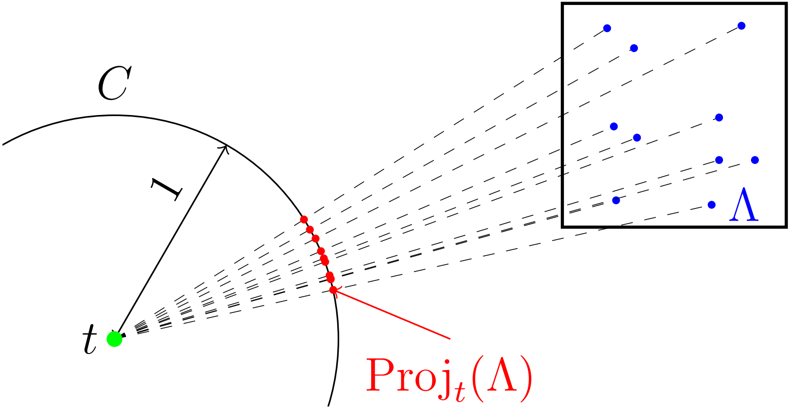
Orthogonal projection to ℓ_θ

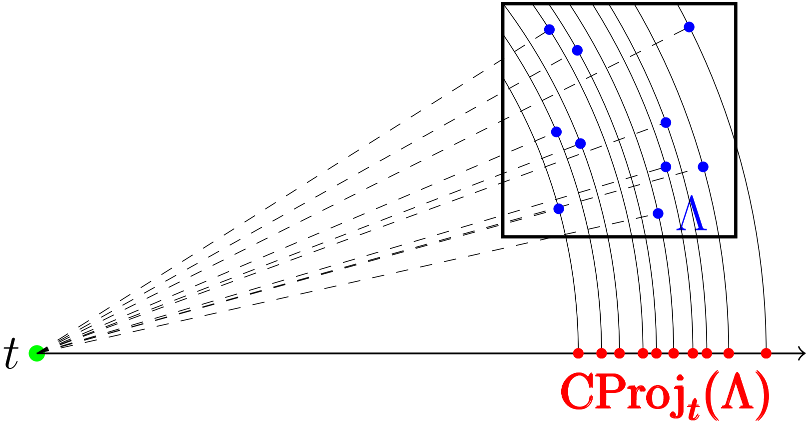


Radial and co-radial projections with center t



Let $\mathrm{CProj}_t(\Lambda) := \{\mathrm{dist}(t, x) : x \in \Lambda\}$ ($\mathrm{CProj}_t(\Lambda)$ is the set of the length of dashed lines above).

The co-radial projection



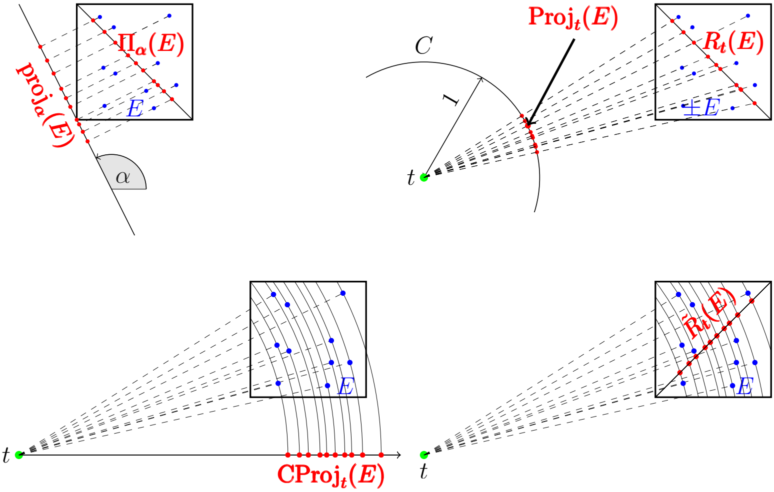


Figure: The orthogonal proj_α, radial Proj_t, co-radial CProj_t projections and the auxiliary projections Π_α, R_t, and $\tilde{R}_t$.

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Λ percolates

Let $\Lambda(\omega)$ be a realization of this random Cantor set. We say that $\Lambda(\omega)$ percolates if there is a connected component of $\Lambda(\omega)$ which connects the left and the right walls of the square $[0,1]^2$.

Let us write $E_{|\leftrightsquigarrow|}$ for the event that the random self-similar set Λ percolates.

Theorem [J.T Chayes, L. Chayes, R. Durrett]

Let TD be the event that Λ is totally disconnected. That is all connected components are singletons. Let

$$p_c := \inf \left\{ p : \mathbb{P}_p \left(E_{|\leftrightsquigarrow|} \right) > 0 \right\}$$

Then $0 < p_c < 1$ and

$$p_c = \sup \left\{ p : \mathbb{P}_p \left(TD \right) = 1 \right\}.$$

If $p < p_c < 1$ then all connected components of Λ are singletons. If $p > p_c$ then Λ percolates with positive probability.

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We assume that

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Then the following statements hold almost surely conditioned on $\Lambda \neq \emptyset$:

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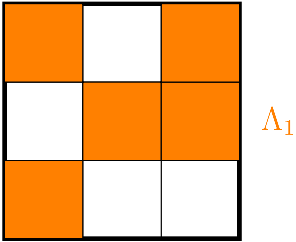

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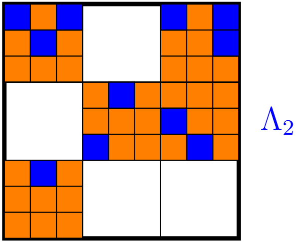

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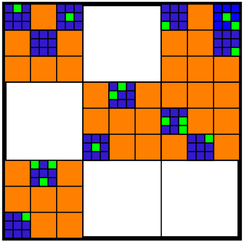

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Theorem Rams, S.

Assume that

1. $\forall k$: $\sum_{i=0}^{M-1} p_{i,k} > 1$ and $\sum_{j=0}^{M-1} p_{k,j} > 1$ and
2. $\forall \alpha \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$, α is good.

Then the following statements hold almost surely conditioned on $\Lambda \neq \emptyset$:

$$\forall \theta \in [0, \pi], \ \mathrm{proj}_{\theta}(\Lambda) \text{ containes an interval }.$$

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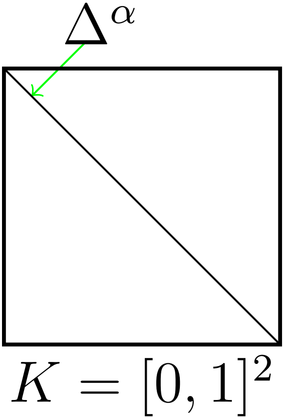
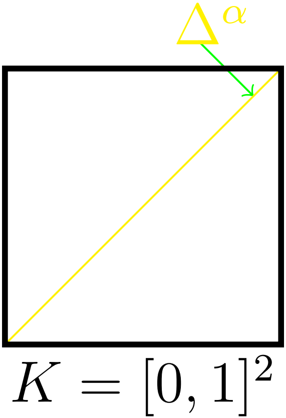
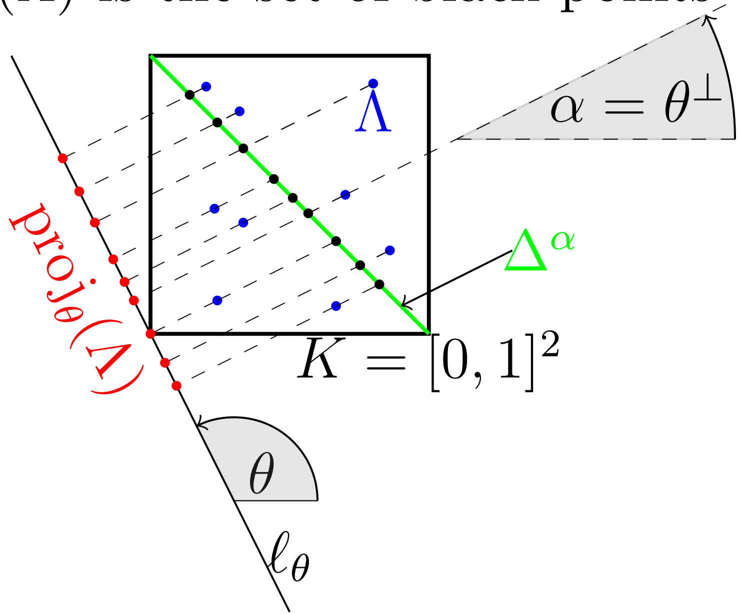
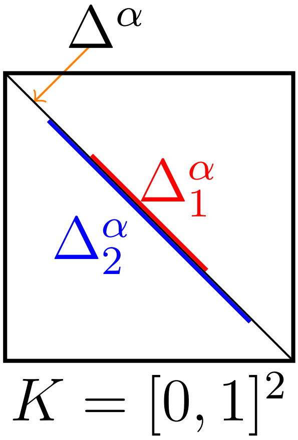




$\Pi_\alpha(\Lambda)$ is the set of black points



α is good if $\exists \Delta_1^\alpha, \Delta_2^\alpha \subset \Delta^\alpha$, and $\exists r_\alpha \in \mathbb{N}$ such that $\Delta_1^\alpha \subset \mathrm{int}(\Delta_2^\alpha)$ and $\forall x \in \Delta_2^\alpha$ the sum of the probabilities of the gray squares > 2.



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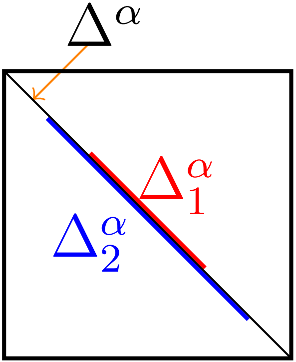

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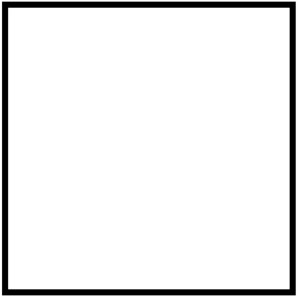

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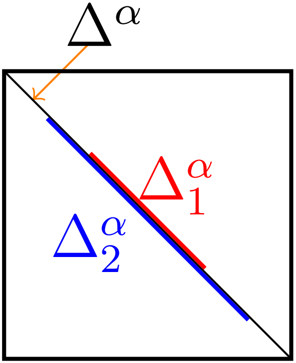

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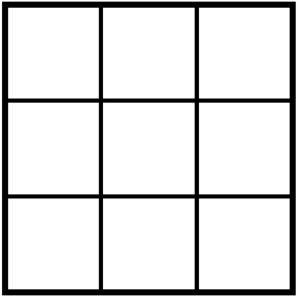

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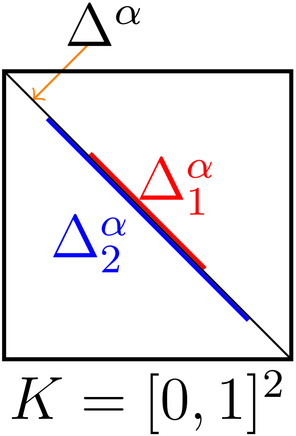

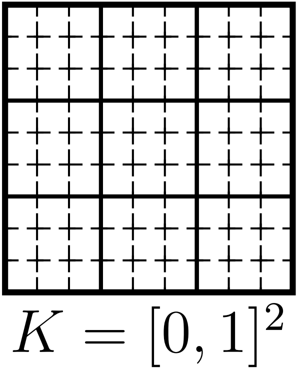

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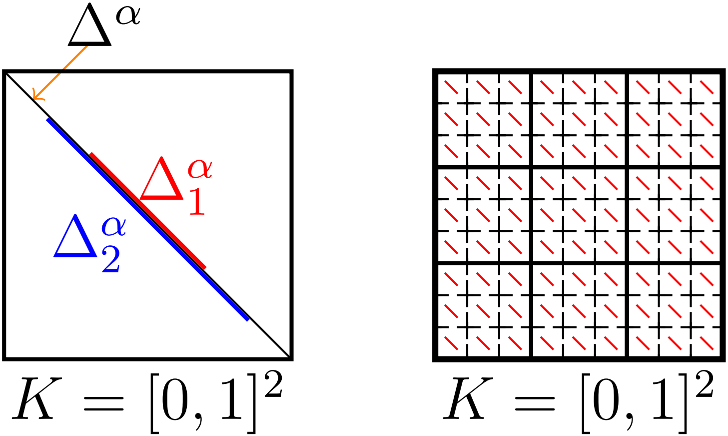

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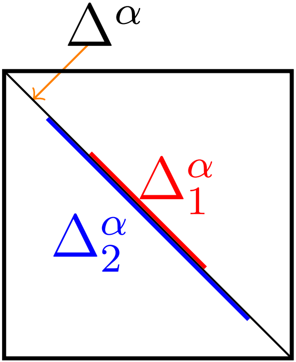

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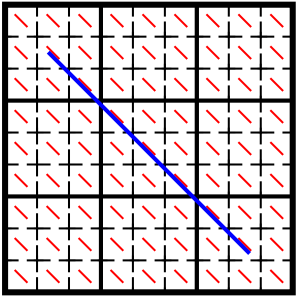

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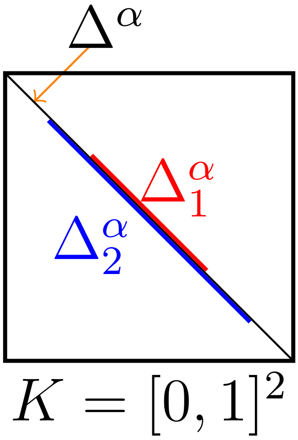

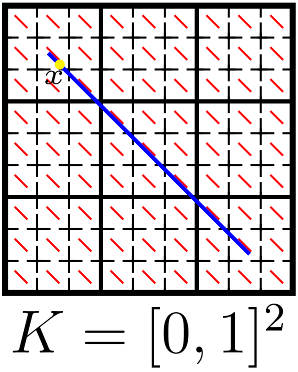

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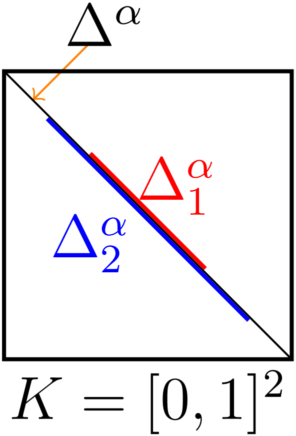

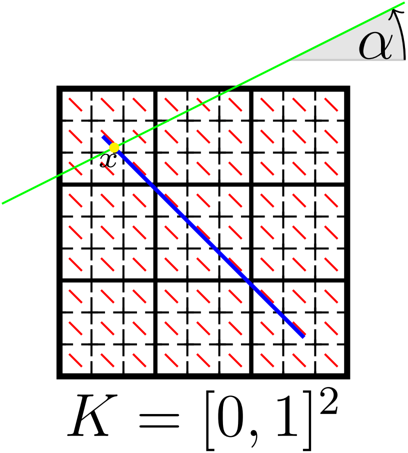

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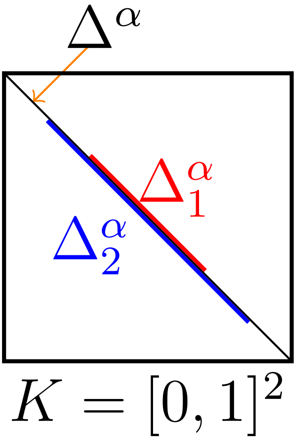

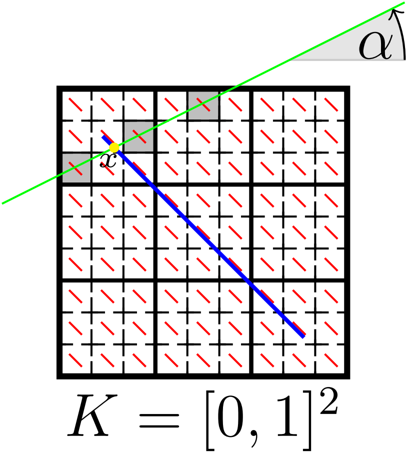

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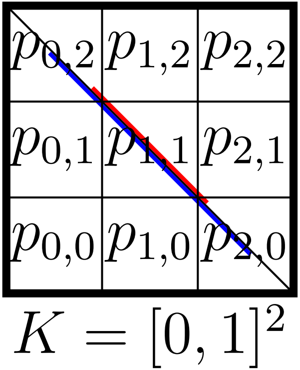

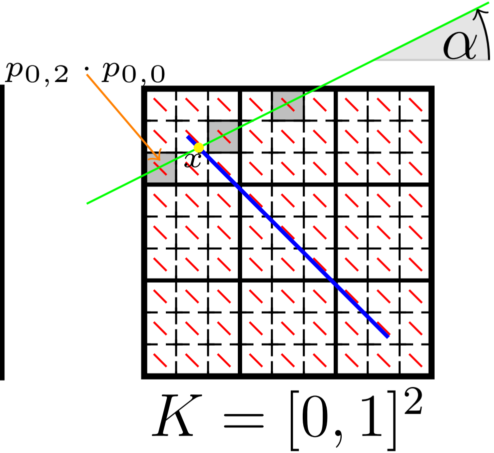

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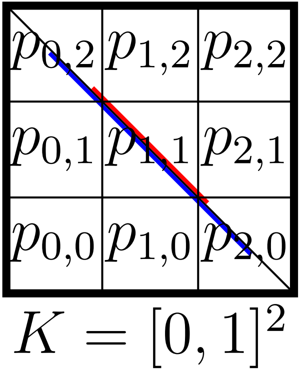

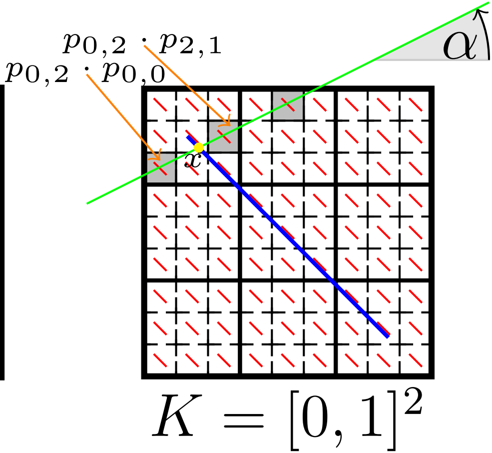

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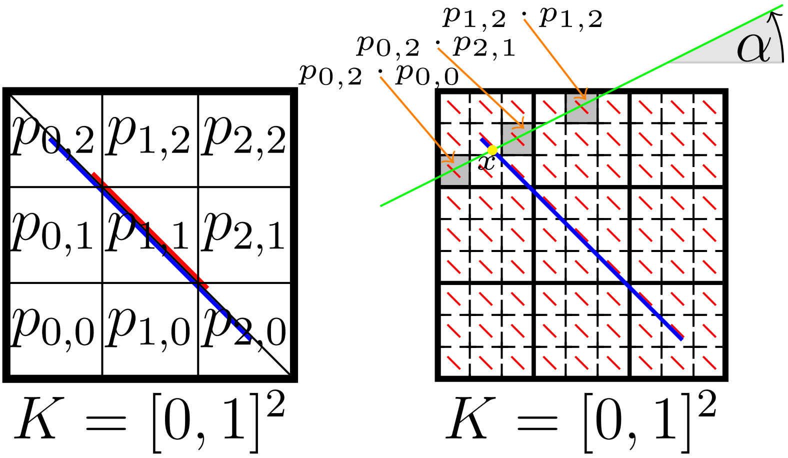

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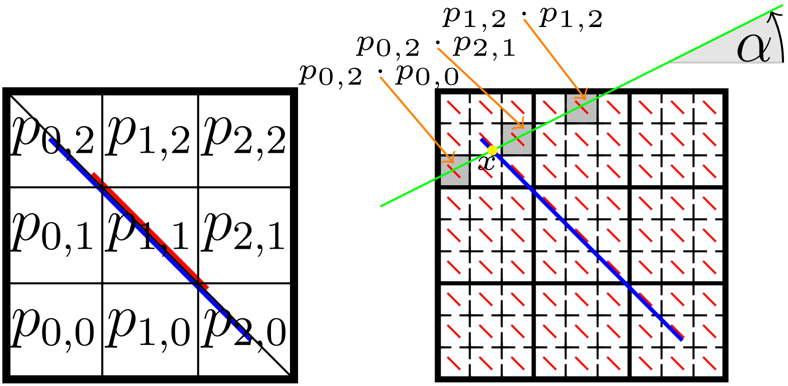
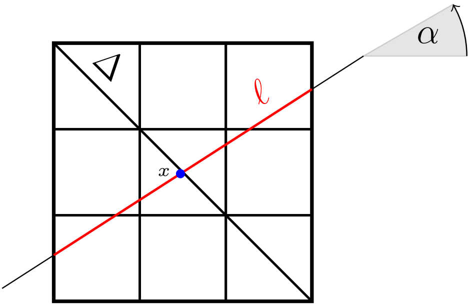


$$p_{0,2} \cdot p_{0,0} + p_{0,2} \cdot p_{2,1} + p_{1,2} \cdot p_{1,2} > 2$$

Remarks

The gray sum is equal to the expected number of level r_α red diagonals whose Π_α-projection covers x.

How to find our if α is a good angle?



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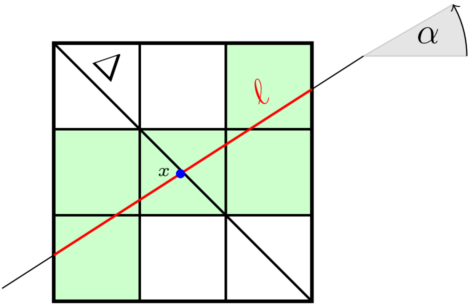

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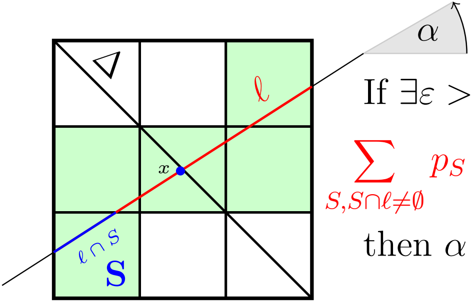
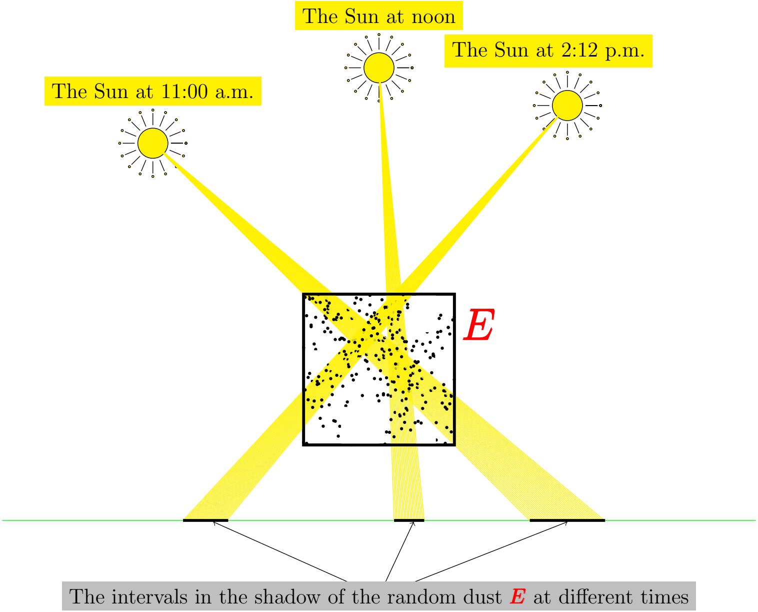


If $\exists \varepsilon > 0$ s.t. $\forall x \in \Delta$:

$$\sum_{S, S\cap\ell\neq\emptyset} p_S \cdot \frac{1}{M} \cdot |\ell \cap S| \geq (1+\varepsilon)\cdot|\ell|$$

then α is a good angle.


The Sun at noon
The Sun at 2:12 p.m.
The Sun at 11:00 a.m.
E
The intervals in the shadow of the random dust E at different times

What happens in dimension higher than 2

Theorem 2 (Vagó and S.)

The same happens in dimension higher than 2 as on the plane.

The method of the proofs is the same in higher dimension. However, there are some technical difficulties that appear in higher dimension which are not present when we work on the plane.

1 History

2 The projections

3 Percolation phenomenon

4 New results

5 Non-homogeneous Fractal percolation sets

6 Homogeneous percolation of small dimension

7 The sum of three linear random Cantor sets

Theorem [Rams, S.] If $\frac{1}{M^2} < p \leq \frac{1}{M}$

Theorem 3

Let $\ell \subset \mathbb{R}^2$ be a straight line and let Λ_ℓ be the orthogonal projection of Λ to ℓ.

*Then for almost all realizations of Λ (conditioned on $\Lambda \neq \emptyset$) and for **all** straight lines ℓ we have:*

$$\dim_{\mathrm{H}}(\Lambda_\ell) = \dim_{\mathrm{H}}(\Lambda). \tag{1}$$

Actually much more is true:

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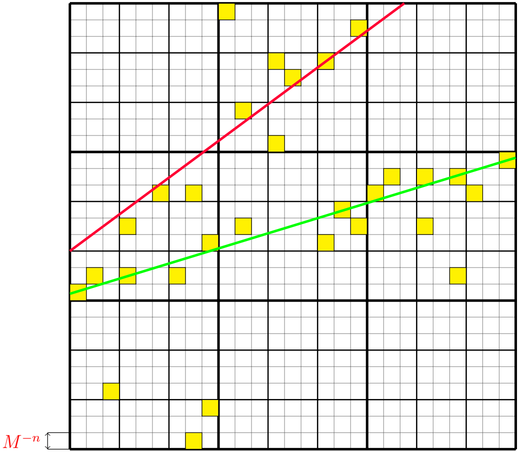
Actually much more is true:

Lines intersect $\leq c \cdot n$ squares of level n

Theorem 4 (Rams, S.)

*If $\frac{1}{M^2} < p \leq \frac{1}{M}$ then for almost all realizations of Λ (conditioned on $\Lambda \neq \emptyset$) and for **all** straight lines ℓ : there exists a constant C such that **the number of level n squares having nonempty intersection with Λ is at most $c \cdot n$**.*
*On the other hand, almost surely for n big enough, we can find **some** line of 45° angle which intersects $\mathtt{const} \cdot n$ level n squares.*

First I draw the theorem and then I state it more precisely.



Recall: 2

$\frac{1}{M^2} < p \leq \frac{1}{M}$ $\Rightarrow$ *Then every line ℓ intersects at most $\mathcal{const} \cdot n$ level n squares.*

Previous theorem stated more precisely I

Recall that Λ_n is the union of retained level-n squares. Let Δ be the decreasing diagonal of the unit square K (the diagonal connecting points $(0,1)$ and $(1,0)$).

Definition 5 (Slices of Λ)

Consider the family of all lines with argument between 0 and $\pi/2$ having non-empty intersection with $\mathrm{int}(\Delta)$. The unit square K cuts out a line segment from each of these lines. Let $\mathfrak{L}$ be the set of all line segments obtained in this way. The sets of the form $\Lambda \cap \ell, \ell \in \mathcal{L}$ are the slices of Λ.

Let $L_n(\ell) := |\Lambda_n \cap \ell|, \quad \ell \in \mathfrak{L}$.

Previous theorem stated more precisely II

Clearly, $\mathfrak{L}$ can be presented as a countable union of families of lines segments $\mathfrak{L}^\theta$ whose angles $\mathrm{Arg}(\ell)$ are θ-separated from both 0 and $\pi/2$:

$$\mathfrak{L}^\theta := \left\{ \ell \in \mathfrak{L} : \min \left\{ \mathrm{Arg}(\ell), \frac{\pi}{2} - \mathrm{Arg}(\ell) \right\} > \theta \right\}, 0 < \theta < \frac{\pi}{4}.$$

Previous theorem stated more precisely II

Corollary 6

For almost all realizations of E we have

$$\forall \theta \in \left(0, \frac{\pi}{4}\right), \ \exists N, \ \forall n \geq N, \ \forall \ell \in \mathfrak{L}^{\theta}; \# \mathcal{E}_n(\ell) \leq \mathrm{const} \cdot n, \tag{2}$$

where $\mathcal{E}_n(\ell)$ *is the number of selected level n squares that intersects* Λ.

Large deviation estimate for $L_n(\ell)$ I

Theorem 7 (Hoeffding)

Let $X_1, \ldots, X_m$ be independent bounded random variables with $a_i \leq X_i \leq b_i$, $(i = 1, \ldots, m)$. Then for any $t > 0$:

$$\mathbb{P}\left(X_1 + \cdots + X_m - \mathbb{E}\left[X_1 + \cdots + X_m\right] \geq t\right) \leq \exp\left(\frac{-2t^2}{\sum_{i=1}^{m}(b_i - a_i)^2}\right).$$

Large deviation estimate for $L_n(\ell)$ II

We apply this to prove:

Lemma 8

For every $u > 1$ *there is a constant* $r = r(u) > 0$ *such that for every* $n \geq 1$, $\ell \in \mathfrak{L}$ *and* $0 < R < |\ell|$,

$$\mathbb{P}\left(L_n(\ell) > pL_{n-1}(\ell) \cdot u \middle| L_{n-1}(\ell) \geq R\right) < \exp\left(-rM^{(n-1)}R\right) \tag{3}$$

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Summary

1. If $0 < p \leq 1/M^2$ then Λ dies out in finitely many steps almost surely.
2. If $\frac{1}{M^2} < p < \frac{1}{M}$ The $\Lambda \neq \emptyset$ with positive probability but $\dim_{\mathrm{H}}(\Lambda) = \frac{\log(M^2 p)}{M} < 1$. For almost all non-empty realizations, for all projections (all radial, co-radial and all orthogonal projections) the dimension of Λ does not decrease under the projection .
3. If $\frac{1}{M} < p < p_c$. Conditioned on non-extinction, almost surely: all projections of Λ contain some intervals but Λ is totally disconnected .
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Definition 9

We say that $f[0,1]^2 \to \mathbb{R}$ is a strictly monotonic smooth function if $f \in \mathcal{C}^2[0,1]$ and $f'_x \neq 0$, $f'_y \neq 0$.

Theorem 10 (Rams, S.)

If $p > \frac{1}{M}$ ($\dim_{\mathrm{H}} \Lambda > 1$) then for every strictly monotonic smooth function f, $f(\Lambda)$ contains an interval, almost surely conditioned on non-extinction.

Examples:

- $\{x + y : (x,y) \in \Lambda\} \supset$ interval .
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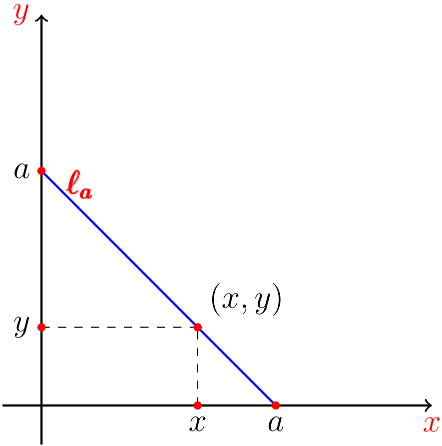


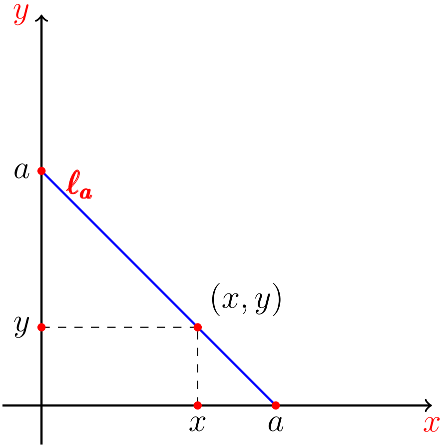
The arithmetic sum of the sets Λ_1, Λ_2 is:

$$\Lambda_1 + \Lambda_2 := \{x + y : x \in \Lambda_1,\ y \in \Lambda_2\}$$

The geometric interpretation of the arithmetic sum is:

$$\Lambda_1 + \Lambda_2 := \{a : \ell_a \cap \Lambda_1 \times \Lambda_2 \neq \emptyset\}.$$

So, $\Lambda_1 + \Lambda_2$ is the 45° projection of $\Lambda_1 \times \Lambda_2$.



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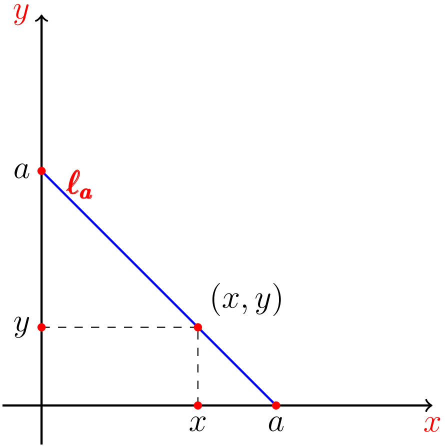

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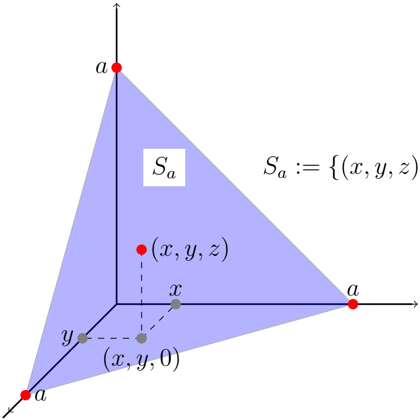


$S_a := \{(x, y, z) : x + y + z = a\}$

$$a = x + y + z \Longleftrightarrow (x, y, z) \in S_a$$

$$\Lambda_1 + \Lambda_2 + \Lambda_3 = \{a : S_a \cap \Lambda_1 \times \Lambda_2 \times \Lambda_3 \neq \emptyset\}.$$

Recall: 4

If $\frac{1}{M^2} < p \leq \frac{1}{M}$ *then for almost all realizations of* Λ *(conditioned on* $\Lambda \neq \emptyset$*) and for* ***all*** *straight lines* ℓ *:* *there exists a constant* C *such that* ***the number of level*** $\boldsymbol{n}$ ***squares having nonempty intersection with*** $\boldsymbol{\Lambda}$ ***is at most*** $\boldsymbol{c \cdot n}$.

The same theorem holds if we substitute the two-dimensional Mandelbrot percolation Cantor set with the product of two independent one dimensional Cantor sets having the same M and probabilities p_1, p_2 such that $p = p_1 \cdot p_2$.

Let $\Lambda_1, \Lambda_2, \Lambda_3$ be one dimensional Mandelbrot percolation fractals constructed with the same M but with may be different probabilities p_1, p_2, p_3. Let Λ be the three dimensional Mandelbrot percolation with the same M and

$$p := p_1 p_2 p_3$$

The random Cantor sets

$$\Lambda_1 \times \Lambda_2 \times \Lambda_3 \text{ and } \Lambda$$

share many common features:

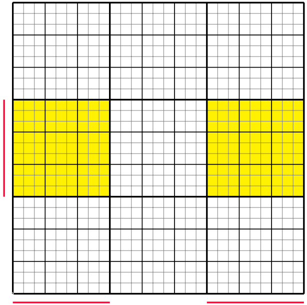
$$\dim \Lambda_1 \times \Lambda_2 \times \Lambda_3 = \dim \Lambda = \frac{\log M^3 p}{\log M}.$$

conditioned on non-extinction.

Dependency in the product set

$$\Lambda_{123} := \Lambda_1 \times \Lambda_2 \times \Lambda_3, \ \Lambda_{12} := \Lambda_1 \times \Lambda_2.$$

In Λ_{123} and in Λ_{12} there is NO independence between the successors of two cubes having one side common.



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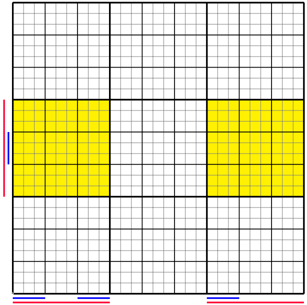

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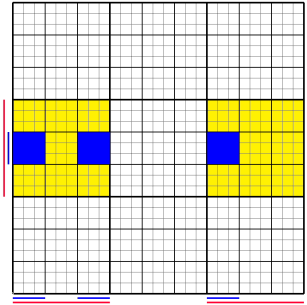

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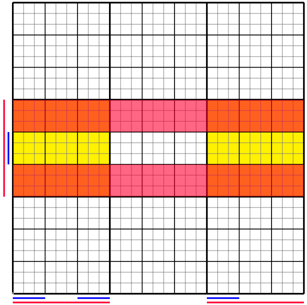

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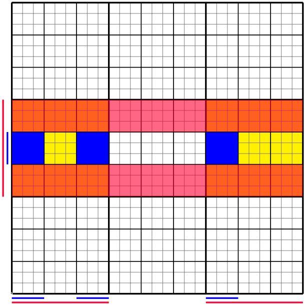
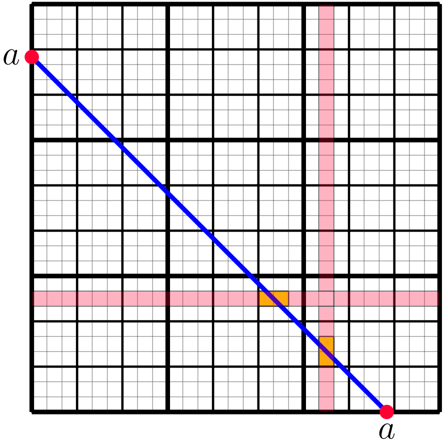


Λ and Λ_{12} are a little bit different from the point of 45° projection

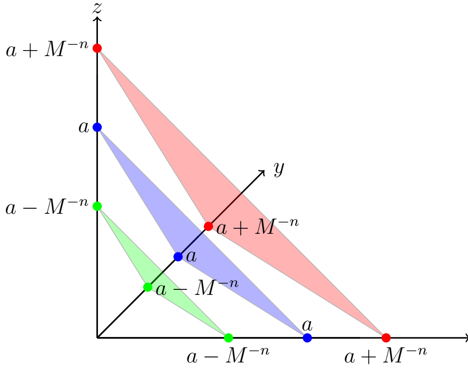


From now we focus on Λ_{123}:

Let $\mathcal{E}^n$ be the set of selected level n cubes in $\Lambda^n_{1,2,3}$.
Since $\dim_{\mathrm{B}} \Lambda_{123} > 1$ so for a $\tau > 0$:

$$\#\mathcal{E}^n \approx M^n \cdot M^{\tau \cdot n}.$$

The colored planes: $3M^n$ planes that are orthogonal to $(1,1,1)$ and the consecutive ones are separated by M^{-n}. By pigeon hole principle one of the planes intersects $\mathrm{const} \cdot M^{\tau n}$ selected level n cubes. Assume that this is the **blue plane**.



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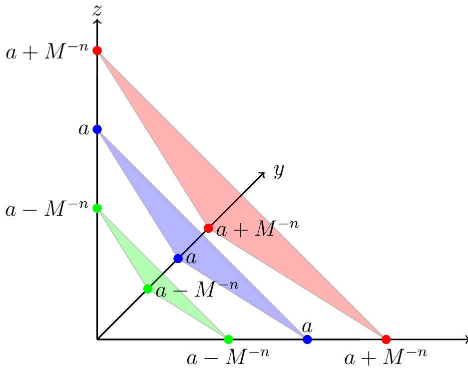

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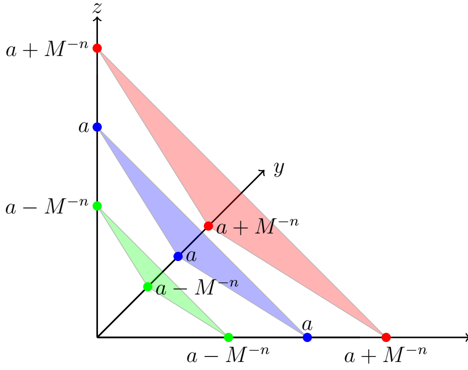
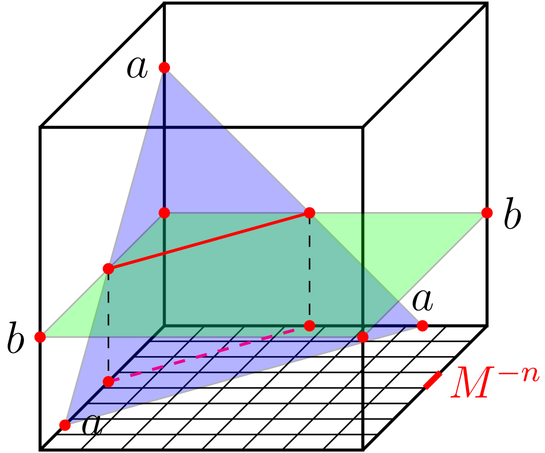


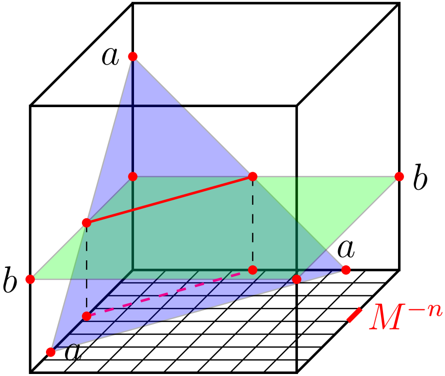
Among the $M^{\tau n}$ cubes which intersect the blue plane the ones sharing one common side are NOT independent.

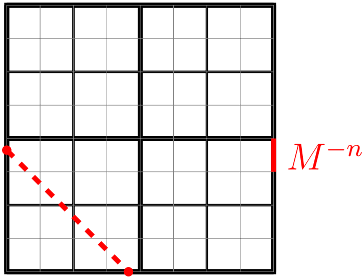
For example those who intersect the red line are NOT independent.



,

$\dim_{\rm H} \Lambda_{123} > 1$ but $\dim_{\rm H} \Lambda_{12}, \dim_{\rm H} \Lambda_{23}, \dim_{\rm H} \Lambda_{31} < 1$.

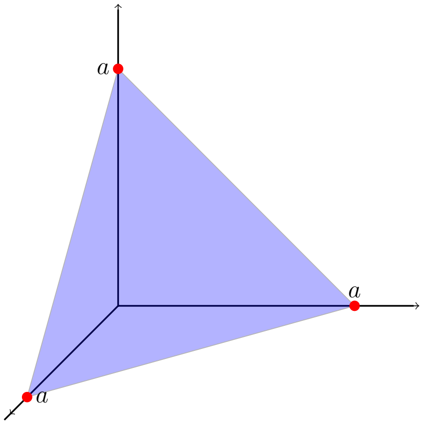




The point is that on the red dashed line there could be potentially M^n selected level n squares but in reality there will be only $c \cdot n$ selected squares.

An easy combinatorial Lemma shows that for a $t > 0$ constant there are M^{nt} selected level n squares that have

- no common sides (so what ever happens in these cubes in the future is independent)
- such that they all intersect the blue plane.



Then we use Large deviation theory similarly to Falconer Grimett to get intervals in the projection.

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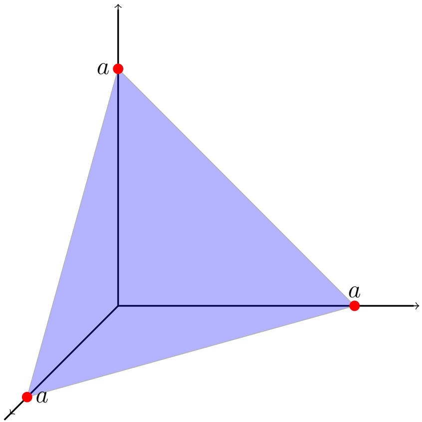

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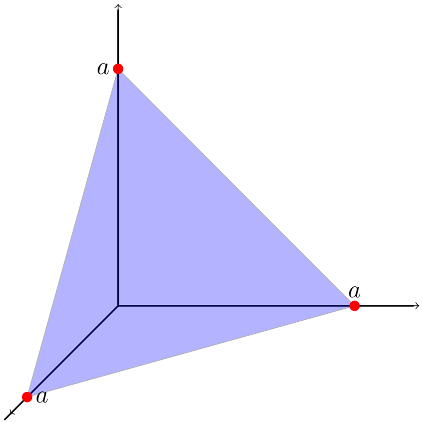

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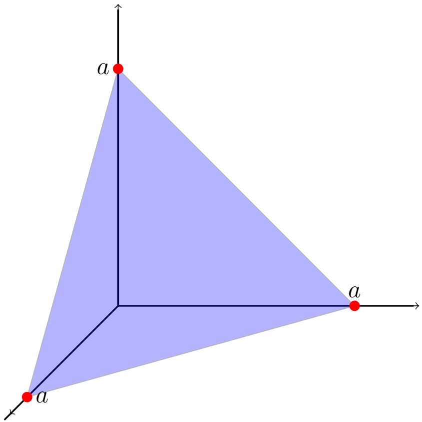

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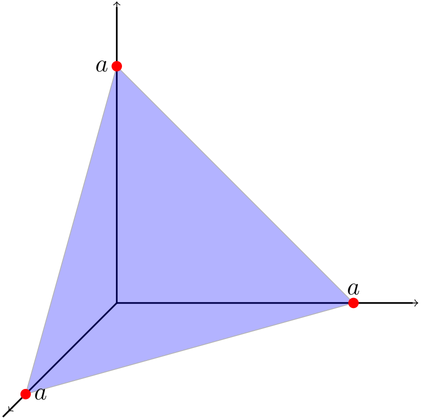

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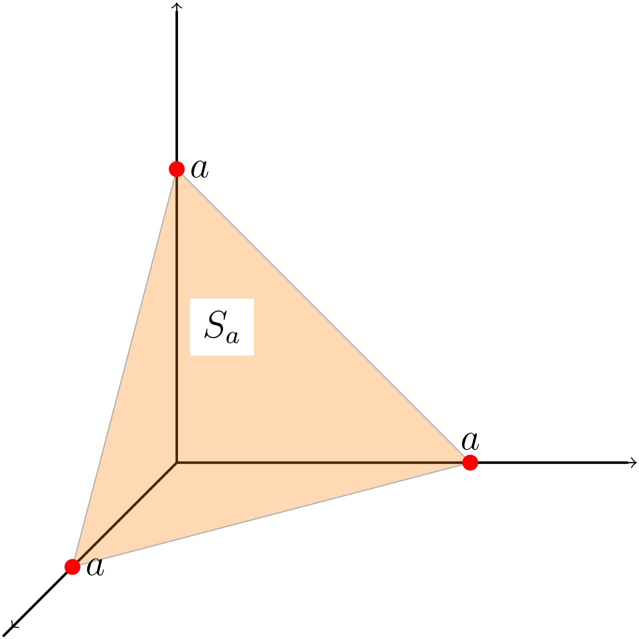
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S_a
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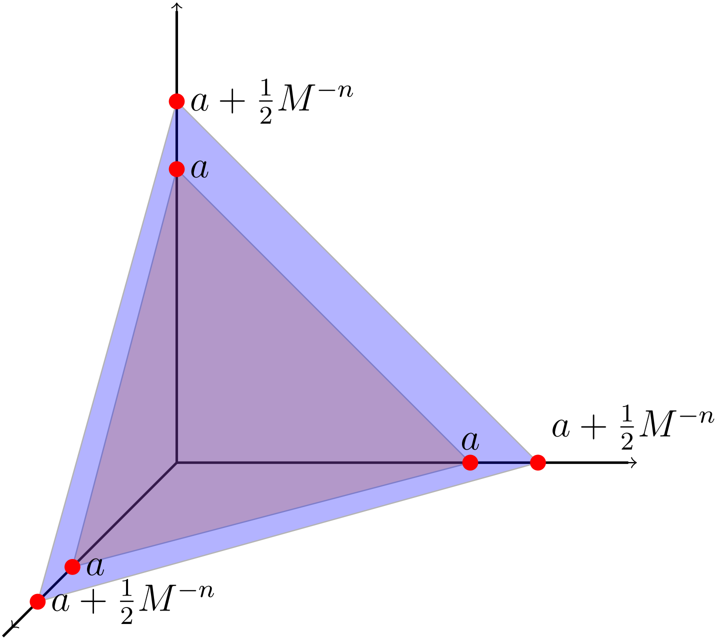
a



$a + \frac{1}{2}M^{-n}$
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$a + \frac{1}{2}M^{-n}$
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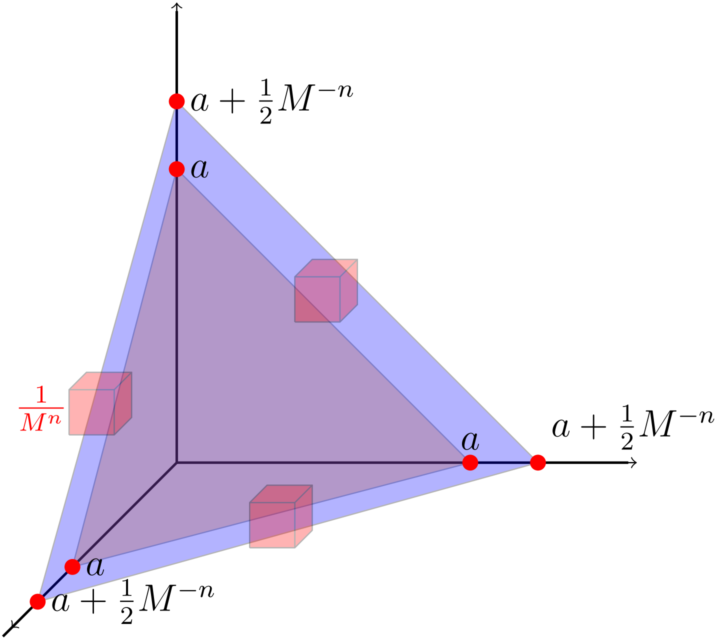

$a + \frac{1}{2}M^{-n}$
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$a + \frac{1}{2}M^{-n}$
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