Analysis 1 B

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These are Lecture Notes for the 1st year Analysis 1B course in Bristol originally prepared by Roman Schubert. This is an evolving version of them, and it is very likely that they still contain misprints. Please report errors and misprints you find to me (`thomas.jordan@bristol.ac.uk`) and I will post an update on the Blackboard page of the course.

These notes cover the main material we will develop in the course, and they are meant to be used parallel to the lectures. The lectures will follow roughly the content of the notes, but sometimes in a different order and sometimes containing additional material. On the other hand, we sometimes refer in the lectures to additional material which is covered in the notes. Besides the lectures and the lecture notes, the homework on the problem sheets is the third main ingredient in the course. Solving problems is the most efficient way of learning mathematics, and experience shows that students who regularly hand in homework do well in the exams.

These lecture notes do not replace a proper textbook in Analysis. Since Analysis appears in almost every area in Mathematics a slightly more advanced textbook which complements the lecture notes will be a good companion throughout your mathematics courses. There is a wide choice of books in the library you can consult.

For the preparation of these notes I mostly consulted

(a) John M. Howie, *Real Analysis*, Springer 2001

(b) Kenneth A. Ross, *Elementary Analysis*, Springer 2013

(c) Michael Reed, *Fundamental Ideas of Analysis*, Wiley 1998

(d) Otto Forster, *Analysis 1*, Springer 2013

The book by Howie focuses on the essentials, and it is the one I followed most closely. For some material which is not covered in the book by Howie, I consulted Ross. The book by Forster is a german textbook, it covers slightly more material than the previous two in a more condensed form. Finally the book by Reed is a very readable introduction to analysis which covers as well some applications and relations to other areas of mathematics and touches on a few more advanced topics.

This course follows on from the Analysis 1A unit, which had 7 chapters, and therefore in these notes we start with Chapter 8.

Chapter 8

Uniform continuity and uniform convergence

In Analysis 1A we considered properties of sequences of real numbers and at the end used these to define continuity of functions. In this first chapter of Analysis 1B we will consider sequences of functions and ask for instance what conditions we need to impose on a sequence of continuous functions, so that the limit is again a continuous function. The two key concepts we introduce in the first two sections are uniform continuity and uniform convergence, respectively.

8.1 Uniform continuity

Let $A \subset \mathbb{R}$, recall that a function $f : A \to \mathbb{R}$ is called continuous on A, if for all $a \in A$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(a)| \leq \varepsilon \quad \text{for all } x \in A \text{ with} \quad |x - a| \leq \delta \ .$$

Here δ depends on ε and on a. If we require δ to be independent of a, then we obtain a stronger notion:

Definition 8.1. *Let $f : A \to \mathbb{R}$, f is called* **uniformly continuous** *on A if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$ we have if $|x - y| \leq \delta$ then $|f(x) - f(y)| \leq \varepsilon$.*

Exercise. *(a) Consider $A = (0, 1)$, $f : (0, 1) \to \mathbb{R}$, $f(x) = 1/x$, this function is continuous, but not uniformly continuous. To see this let us choose an $\varepsilon > 0$ and assume there exists a $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ for $|y - x| \leq \delta$. Then in particular we should have $f(x) - f(x + \delta) \leq \varepsilon$ for all $x \in (0, 1 - \delta)$, but*

$$f(x) - f(x + \delta) = \frac{1}{x} - \frac{1}{x + \delta} = \frac{\delta}{x(x + \delta)}$$

and the right hand side becomes arbitrary large for $x \to 0$.

(b) *Consider* $A = [1/2, 1)$, $f : [1/2, 1) \to \mathbb{R}$, $f(x) = 1/x$ *then* f *is uniformly continuous. Since* $1/x \leq 2$ *for* $x \in [1/2, 1)$ *we have*

$$|f(x) - f(y)| = \frac{|y - x|}{yx} \leq 4|y - x| \leq \varepsilon$$

for $|y - x| \leq \delta$ *with* $\delta = \varepsilon/4$.

A continuous function on a closed bounded interval turns out to be uniformly continuous, this is the main result of this section.

Theorem 8.2. *Let* $f : [a, b] \to \mathbb{R}$ *be continuous, then* f *is uniformly continuous on* $[a, b]$.

Proof. Let us assume f is not uniformly continuous, then there exists an $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there are $x_n, y_n \in [a, b]$ such that

$$|x_n - y_n| \leq \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(y_n)| > \varepsilon \ .$$

This is the negation of the condition for uniform continuity, where $1/n = \delta$. Now (x_n) is a bounded sequence, hence by the Theorem of Bolzano Weierstrass there exists a convergent subsequence (x_{n_k}) with $\lim_{k\to\infty} a_{n_k} = p$, and since $a \leq a_{n_k} \leq b$ we have $p \in [a, b]$ (here we used that the interval $[a, b]$ is closed, otherwise the limit might not lie in the interval). Since $|x_n - y_n| \leq \frac{1}{n}$ the subsequence (y_{n_k}) converges as well to p and since f is continuous on $[a, b]$ we have

$$\lim_{n\to\infty} |f(y_{n_k}) - f(x_{n_k})| = 0$$

which contradicts the assumption that $|f(x_n) - f(y_n)| > \varepsilon$. Hence f is uniformly continuous. □

Example. *Discretisation. Let us discuss the problem of how to approximate a function* $f : [a, b] \to \mathbb{R}$ *by finitely many values. Define for* $n \in \mathbb{N}$ $x_k := a + \frac{k}{n}(b - a)$, $k = 1, 2, \cdots, n$, *then we can consider the values of* f *at the points* x_k,

$$(f(x_1), f(x_2), \cdots, f(x_n)) \in \mathbb{R}^n \ , \tag{8.1}$$

as providing a discrete approximation for f. *We can define a step function* $\psi_n(x) : [a, b] \to \mathbb{R}$ *by*

$$\psi_n(x) := f(x_k) \ , \quad \textit{for} \quad x \in (x_{k-1}, x_k] \tag{8.2}$$

and $\psi_n(a) := f(a)$, *see Figure 8.1. The following result tells us that this construction provides a good approximation if the function* f *is uniformly continuous.*

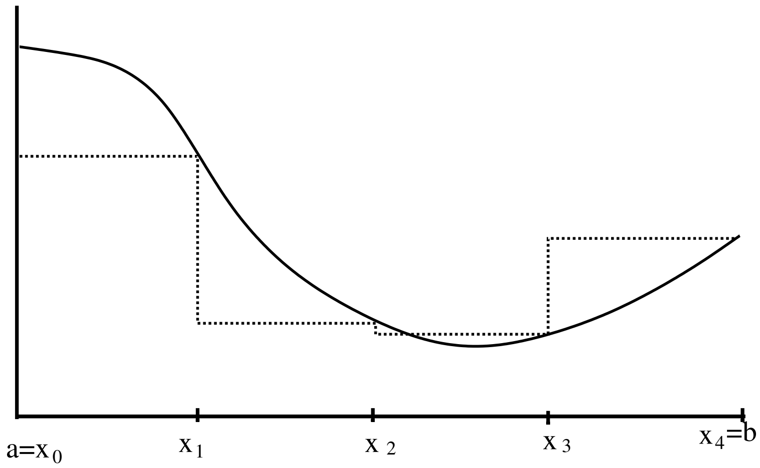

Figure 8.1: Sketch of the approximation of a uniformly continuous function $f(x)$ (solid line) by a step function $\psi(x)$ (dashed line) in Example 8.1.

Theorem 8.3. *Suppose $f : [a, b] \to \mathbb{R}$ is uniformly continuous and $\psi_n(x)$, $n \in \mathbb{N}$, is constructed as in Example 8.1, then for any $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$*

$$|f(x) - \psi_n(x)| \leq \varepsilon \quad \textit{for all} \quad x \in [a, b] \ . \tag{8.3}$$

Proof. Let $\varepsilon > 0$, since f is uniformly continuous, there exists a $\delta > 0$ such that $|f(x) - f(x_k)| \leq \varepsilon$ if $|x - x_k| \leq \delta$. Now because $x_k - x_{k-1} = (b - a)/n$ we see that if $N > (b - a)/\delta$, then $|f(x) - \psi_n(x)| \leq \varepsilon$. □

Example. *The conclusion of the theorem is no longer true if we just assume continuity. Consider the function $f : (0, 1] \to \mathbb{R}$, $f(x) := 1/x$, then for any $n \in \mathbb{N}$*

$$\sup_{x\in(0,1/n]} \left| \frac{1}{x} - n \right| = \infty \ , \tag{8.4}$$

hence we cannot approximate this function by the ψ_n defined in (8.3) *uniformly for all $x \in (0, 1]$. The precise meaning of this statement will be discuss in the next section.*

8.2 Pointwise and uniform convergence

Let us now consider sequences of functions (f_n), where $f_n : A \to \mathbb{R}$. It turns out that one can introduce many different notions of convergence for sequences of functions, which depend on what properties one wants the limits to have. The first notion of convergence one would introduce is to ask for $(f_n(x))$ to be convergent for any $x \in A$, this is called pointwise convergence:

Definition 8.4. *Let $(f_n)_{n\in\mathbb{N}}$, where $f_n : A \to \mathbb{R}$, be a sequence of functions, we say that f_n converges to $f : A \to \mathbb{R}$* **pointwise** *if for any $x \in A$ the sequence $(f_n(x))$ converges to $f(x)$, or, if for any $\varepsilon > 0$ and $x \in A$, there exists $N \in \mathbb{N}$ such that*

$$|f_n(x) - f(x)| \leq \varepsilon \quad \textit{for all} \quad n \geq N \ .$$

Exercise. *(a) Let $f_n(x) := x^n : [0,1] \to \mathbb{R}$. Then*

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases} \ .$$

(b) Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} -1 & x \leq -\frac{1}{n} \\ nx & -\frac{1}{n} < x < \frac{1}{n} \\ 1 & x \geq \frac{1}{n} \end{cases} \ ,$$

then $\lim_{n\to\infty} f_n(x) = \chi(x)$ where $\chi(x) = x/|x|$ for $x \neq 0$ and $\chi(0) = 0$.

(c) Let $g_n : \mathbb{R} \to \mathbb{R}$ be

$$g_n(x) := \frac{x}{\sqrt{1/n + x^2}} \ ,$$

then $\lim_{n\to\infty} g_n(x) = \chi(x)$.

The examples show that even if the functions in a sequence f_n are continuous, the limit does not have to be continuous.

So in order to guarantee that a limit of a sequence of continuous functions is continuous we have to pose stronger conditions on the mode of convergence.

Definition 8.5. *Let $(f_n)_{n\in\mathbb{N}}$, where $f_n : A \to \mathbb{R}$, be a sequence of functions, we say that f_n converges to $f : A \to \mathbb{R}$* **uniformly** *if for any $\varepsilon > 0$ there exist a $N \in \mathbb{N}$ such that*

$$|f_n(x) - f(x)| \leq \varepsilon \quad \textit{for all} \quad n \geq N \ \textit{and} \ x \in A \ .$$

The relationship between pointwise convergence and uniform convergence is similar to the relationship between continuity and uniform continuity. In the definition of pointwise convergence the N depends on $\varepsilon > 0$ and $x \in A$, whereas in the definition of uniform convergence the N is independent of x. Hence $\lim_{n\to\infty} f_n = f$ uniformly means for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$

$$\sup_{x\in A} |f_n(x) - f(x)| \leq \varepsilon \ .$$

Exercise. *(a) The sequence in first example above,* $f_n(x) := x^n : [0,1] \to \mathbb{R}$*, does not converge uniformly. Let* $p \in (0,1)$ *and set* $x_n = p^{1/n}$*, then* $f_n(x_n) = p$*, so*

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| \geq p$$

independent of n*.*

(b) If we restrict in the previous example the interval to $[0,a]$ *with* $a \in (0,1)$*, then*

$$\sup_{x\in[0,a]} |f_n(x)| = a^n$$

and since $\lim_{n\to\infty} a^n = 0$ *there exist for any* $\varepsilon > 0$ *a* $N \in \mathbb{N}$ *such that* $\sup_{x\in[0,a]}|f_n(x)| \leq \varepsilon$ *for* $n \geq N$*, so* $\lim_{n\to\infty} f_n(x) = 0$ *uniformly on* $[0,a]$ *if* $0 < a < 1$*.*

(c) Consider the sequence $f_n : [0,1] \to \mathbb{R}$*,*

$$f_n(x) = \frac{x}{1 + (x/2)^n} .$$

Since $a^n \to 0$ *for* $n \to \infty$ *if* $|a| < 1$ *we have for any* $x \in [0,1]$ *that* $\lim_{n\to\infty} f_n(x) = x$ *pointwise. Now*

$$|f_n(x) - x| = x\left|\frac{1 - (1 + (x/2)^n)}{1 + (x/2)^n}\right| = x\frac{(x/2)^n}{1 + (x/2)^n} \leq x(x/2)^n \leq 1/2^n$$

so for any $\varepsilon > 0$ *there exist a* $N \in \mathbb{N}$ *such that* $|f_n(x) - x| \leq \varepsilon$ *for* $n > N$ *since* $1/2^n \to 0$ *for* $n \to \infty$*. So we have as well* $\lim_{n\to\infty} f_n = x$ *uniformly.*

We can now show that a uniform limit of a sequence of continuous functions is continuous.

Theorem 8.6 (Weierstrass)**.** *Let* $f_n : [a,b] \to \mathbb{R}$ *be a sequence of functions which converge uniformly to* $f : [a,b] \to \mathbb{R}$*. If the functions* f_n *are continuous, then* f *is continuous.*

Proof. Let $\varepsilon > 0$, since (f_n) converges uniformly to f there exists $N \in \mathbb{N}$ such that for $n > N$ and all $x \in [a,b]$

$$|f(x) - f_n(x)| \leq \varepsilon/3 .$$

Fix an n with $n > N$, since f_n is uniformly continuous there exists $\delta > 0$ such that

$$|f_n(x) - f_n(y)| \leq \varepsilon/3$$

for $y \in (x-\delta, x+\delta) \cap [a,b]$, and so for $y \in (x-\delta, x+\delta) \cap [a,b]$

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

Hence f is continuous on $[a,b]$. □

The notion of uniform convergence is often formalised by using a norm.

Definition 8.7. *Let $A = [a,b]$ be a closed interval, and $f : [a,b] \to \mathbb{R}$ be bounded, the* **supremum norm** *(or sup norm) of f is defined as*

$$\|f\| := \sup_{x \in A} |f(x)|$$

The sup norm measure the size of a function in terms of its maximum value. The following properties mean that it is a norm on the vector space of all bounded functions on A.

Theorem 8.8. *Let $f, g : A \to \mathbb{R}$ be bounded and $\lambda \in \mathbb{R}$, then we have*

(i) $\|f\| \geq 0$ *and* $\|f\| = 0$ *if and only if* $f = 0$.

(ii) $\|\lambda f\| = |\lambda| \|f\|$

(iii) $\|f + g\| \leq \|f\| + \|g\|$

We will leave this as an exercise. We remark that the example of norms we encounter typically in linear algebra are connected with an inner product. The sup norm is an example of a norm which is not connected to an inner product.

In terms of the norm the condition that sequence of functions (f_n), $f_n : [a,b] \to \mathbb{R}$ converges uniformly to f, can be expressed as

$$\lim_{n \to \infty} \|f_n - f\|_{[a,b]} = 0 \ .$$

Definition 8.9. *We denote by $C([a,b])$ the set of all (uniformly) continuous functions on $[a,b]$. We say that (f_n), where $f_n \in C([a,b])$ for all $n \in \mathbb{N}$, is a Cauchy sequence in $C([a,b])$ if for any $\varepsilon > 0$ there exist a $N \in \mathbb{N}$ such that*

$$\|f_n - f_m\|_{[a,b]} \leq \varepsilon \quad \text{for all} \quad m, n > N \ .$$

Uniformly convergent sequences are always Cauchy.

Proposition 8.10. *Let (f_n) be a sequence of functions in $C([a,b])$ and $f \in C([a,b])$ such that $\lim_{n\to\infty} \|f_n - f\|_{[a,b]} = 0$. In this case (f_n) is a Cauchy sequence.*

Proof. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that if $n \geq N$ then $\|f_n - f\|_{[a,b]} \leq \varepsilon/2$. If $n, m \geq N$ then using part (iii) from the previous theorem (triangle inequality)

$$\|f_n - f_m\|_{[a,b]} = \|f_n - f + f - f_m\|_{[a,b]} \leq \|f_n - f\|_{[a,b]} + \|f - f_m\|_{[a,b]} \leq \varepsilon.$$

□

Cauchy sequences are of great importance in Analysis. It is always true that convergent sequences are Cauchy and if a set has the property that all Cauchy sequences are convergent with the limit contained in the set, it is called complete. An example is the set of real numbers $\mathbb{R}$. Another example is the set of continuous functions on $[a, b]$.

Theorem 8.11. *Let (f_n) be a Cauchy sequence in $C([a, b])$, then there exists an $f \in C([a, b])$ such that $\lim_{n\to\infty} f_n = f$.*

Proof. Let us first fix an $x \in [a, b]$, then $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_{[a,b]}$, hence the sequence of real numbers $(f_n(x))$ is a Cauchy sequence, and therefore there exist a limit $f(x)$. This limit exist for any $x \in [a, b]$ and so there exist a function $f : [a, b] \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n = f$ pointwise.

Let us now assume that the convergence is not uniform, i.e, there exist a $\varepsilon > 0$ and a sequence $x_n \in [a, b]$ such that

$$|f_n(x_n) - f(x_n)| \geq 2\varepsilon \ .$$

Since (f_n) is a Cauchy sequence there exists an $N \in \mathbb{N}$ such that $|f_n(x_n) - f_m(x_n)| \leq \varepsilon$ for all $m, n > N$, and let us fix one such n. Now by the triangle inequality $|f_n(x_n)-f(x_n)| \leq |f_n(x_n)-f_m(x_n)|+|f_m(x_n)-f(x_n)|$ and hence

$$|f_n(x_n) - f_m(x_n)| \geq 2\varepsilon - |f_m(x_n) - f(x_n)| \ .$$

Since f_m converges pointwise to f, there exist an $m > N$ such that $|f_m(x_n)-f(x_n)| \leq \varepsilon$, and therefore

$$|f_n(x_n) - f_m(x_n)| \geq \varepsilon \ ,$$

which contradicts the fact that f_n is a Cauchy sequence. So the convergence must be uniform and then by Weierstrass' Theorem $f \in C([a, b])$. □

The importance of results of this type comes from the fact that in mathematics, both in pure and in applied areas, one often constructs solutions to a problem in an iterative way, starting with a first guess f_1 then improving it to second guess f_2 and so on. In this way one creates a sequence of approximate solutions f_n and what one knows usually from the construction is that the difference between them will get smaller and smaller with larger n, which is the Cauchy property. The limit, if it exists, is then supposed to be the solution to the problem, and a Theorem of the type above tells us that such a limit exists and has good properties.