Topics in Algebraic Geometry assignments.

Problem 1 (for 29/1). Take the following curves in $\mathbb{A}^2_{x,y}$

$$C : y^2 = x^3, \qquad D : y^2 = x^3 + x^2, \qquad E : y^2 = x^3 + x,$$

Prove that the completed local rings at $p = (0, 0)$ are

$$\hat{\mathcal{O}}_{C,p} \cong k[[t^2, t^3]], \qquad \hat{\mathcal{O}}_{D,p} \cong k[[s, t]]/st, \qquad \hat{\mathcal{O}}_{E,p} \cong k[[t]],$$

and that they are pairwise non-isomorphic when $\operatorname{char} k \neq 2$.

Problem 2 (for 5/2).

(1) Show that $\mathbb{P}^1 \times \mathbb{P}^1 \not\cong \mathbb{P}^2$. (You may want to use 'weak Bezout'.)

(2) If V is any variety, a rational map $f : V \rightsquigarrow \mathbb{P}^n$ is given by $n + 1$ rational functions $f_0, ..., f_n \in k(V)$ (not all identically zero on V),

$$V \ni P \quad \longmapsto \quad [f_0(P) : ... : f_n(P)] \in \mathbb{P}^n,$$

and $gf_0, ..., gf_n$ give the same map, for $g \in k(V)^\times$. If, for a point $P \in V$, there is such a g that the gf_i are all defined and not all zero at P, we say that f is regular (or defined) at P, and $f(P)$ is the corresponding value. Use this to show that $\mathbb{P}^n$ is complete, by verifying the valuative criterion.

Problem 3 (for 12/2). Suppose C/k is a complete non-singular curve that admits a map $x : C \to \mathbb{P}^1$ of degree 2, in other words C is hyperelliptic. For simplicity, assume $\operatorname{char} k = 0$.

(1) Show that C is birational to a curve $y^2 = f(x) \subset \mathbb{A}^2$, with $f \in k[x]$ square-free. (Hint: Describe $k(C)$.)

(2) Conversely, if $f(x) \in k[x]$ is squarefree, of degree $2g + 1$ or $2g + 2$, for some $g > 0$, the two affine charts

$$y^2 = f(x) \qquad \text{and} \qquad Y^2 = X^{2g+2} f(\tfrac{1}{X})$$

glue via $Y = \frac{y}{x^{g+1}}$, $X = \frac{1}{x}$ to a complete, non-singular curve C. (You may use this.) Show that C has genus g, with regular differentials

$$\Omega_C = \Big\langle \frac{dx}{y}, \frac{x\,dx}{y}, \ldots, \frac{x^{g-1}dx}{y} \Big\rangle.$$

Now let C be *any* complete non-singular curve of genus 2. Use $\deg K_C = 2$ and $\dim \mathcal{L}(K_C) = 2$ to prove that C is hyperelliptic.

Problem 4 (for 19/2). Suppose C/k ($\operatorname{char} k \neq 2$) is a hyperelliptic curve of genus $g \geq 1$, given by an equation

$$y^2 = x^{2g+1} + a_{2g}x^{2g} + \ldots + a_0.$$

Write ∞ for the unique point at infinity of C.

(1) Use Cantor's description of divisors to describe the 2-torsion elements (elements of order 2) in $\operatorname{Pic}^0(C)$. Show that they form a group $\cong \mathbb{F}_2^{2g}$, and describe how to add them explicitly.

(2) Suppose $P \in C \cap \mathbb{A}^2$ is a 'torsion point of order $2g+1$', in the sense that the divisor $D = (P) - (\infty)$ is $(2g+1)$-torsion,

$$(2g+1)D \sim 0.$$

E.g. by considering the function $f \in \mathcal{L}((2g+1)(\infty))$, that defines the latter equivalence, its image under the hyperelliptic involution, and the natural basis of $\mathcal{L}((2g+1)(\infty))$, show that C has an equation of the form

$$y^2 = x^{2g+1} + (b_g x^g + \ldots + b_1 x + b_0)^2.$$

(This illustrates the fact that high-order torsion points on curves are rare.)

Problem 5 (for 26/2).

(1) Prove that $\operatorname{Aut}\mathbb{P}^1 \cong \mathrm{PGL}_2(k)$.
(2) Find $\operatorname{Aut}\mathbb{A}^1$ and $\operatorname{Aut}(\mathbb{A}^1 \setminus \{0\})$.
(3) Find $\operatorname{Aut}\mathbb{G}_m$ (isomorphisms $\mathbb{G}_m \to \mathbb{G}_m$ as an algebraic group).

Problem 6 (for 3/3).

(1) Prove that over $K = \mathbb{R}$, the unit circle group $S^1 : x^2 + y^2 = 1$ is the only non-trivial form of $\mathbb{G}_m$ up to isomorphism (as algebraic groups).
(2) Similarly, over $K = \mathbb{F}_p$, prove that $\mathbb{G}_m$ has a unique non-trivial form. Write it down an an algebraic group (equations + structure morphisms), and determine its number of points over K.

Problem 7 (for 10/3). In any category $\mathcal{C}$, we can define a 'group object' as one for which the functor

$$\operatorname{Hom}(-, X) : \; \mathcal{C} \longrightarrow \textbf{Sets}$$

factors through the category of groups. Prove that group objects in the category of varieties are (connected) algebraic groups, as we defined them.