Profinite number theory

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The factorial number system

Each \( n \in \mathbb{Z}_{\geq 0} \) has a unique representation

\[
  n = \sum_{i=1}^{\infty} c_i i! \quad \text{with} \quad c_i \in \mathbb{Z},
\]

\[
  0 \leq c_i \leq i, \quad \# \{ i : c_i \neq 0 \} < \infty.
\]

In factorial notation:

\[
  n = (\ldots c_3 c_2 c_1)!.
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In factorial notation:

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\]

*Examples:* \( 25 = (1001)! \), \( 1001 = (121221)! \).

*Note:* \( c_1 \equiv n \mod 2 \).
Conversion

Given $n$, one finds all $c_i$ by

$$c_1 = \text{(remainder of } n_1 = n \text{ upon division by } 2),$$
$$c_i = \text{(remainder of } n_i = \frac{n_{i-1} - c_{i-1}}{i} \text{ upon division by } i+1),$$

until $n_i = 0$. 

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until $n_i = 0$.

Knowing $c_1, c_2, \ldots, c_{k-1}$ is equivalent to knowing $n$ modulo $k!$. 
Profinite numbers

If one starts with $n = -1$, one finds $c_i = i$ for all $i$:

$$-1 = (\ldots54321)!.$$

In general, for a negative integer $n$ one finds $c_i = i$ for almost all $i$. 
Profinite numbers

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$$-1 = (\ldots 54321)!.$$ 

In general, for a negative integer $n$ one finds $c_i = i$ for almost all $i$.

A profinite integer is an infinite string $(\ldots c_3 c_2 c_1)!$ with each $c_i \in \mathbb{Z}$, $0 \leq c_i \leq i$.

Notation: $\hat{\mathbb{Z}} = \{\text{profinite integers}\}$. 
A citizen of the world

Features of $\hat{\mathbb{Z}}$: 

• it has an algebraic structure,  
• it comes with a topology,  
• it occurs in Galois theory,  
• it shows up in arithmetic geometry,  
• it connects to ultrafilters,  
• it carries “analytic” functions,  
• and it knows Fibonacci numbers!
Addition and multiplication

For any $k$, the $k$ last digits of $n + m$ depend only on the $k$ last digits of $n$ and of $m$.

Likewise for $n \cdot m$. 
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For any $k$, the $k$ last digits of $n + m$ depend only on the $k$ last digits of $n$ and of $m$.

Likewise for $n \cdot m$.

Hence one can also define the sum and the product of any two profinite integers, and $\hat{\mathbb{Z}}$ is a commutative ring.
Ring homomorphisms

Call a profinite integer \((\ldots c_3 c_2 c_1)!\) even if \(c_1 = 0\) and odd if \(c_1 = 1\).

The map \(\hat{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z}, (\ldots c_3 c_2 c_1)! \mapsto (c_1 \mod 2),\) is a ring homomorphism. Its kernel is \(2\hat{\mathbb{Z}}\).
Ring homomorphisms

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The map $\hat{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z}$, $\ldots c_3c_2c_1)! \mapsto (c_1 \mod 2)$, is a ring homomorphism. Its kernel is $2\hat{\mathbb{Z}}$.

More generally, for any $k \in \mathbb{Z}_{>0}$, one has a ring homomorphism $\hat{\mathbb{Z}} \to \mathbb{Z}/k!\mathbb{Z}$ sending $\ldots c_3c_2c_1)!$ to $(\sum_{i<k} c_ii! \mod k!)$, and it has kernel $k!\hat{\mathbb{Z}}$. 
Visualising profinite numbers

Define $v : \hat{\mathbb{Z}} \to [0, 1]$ by

$$v((\ldots c_3 c_2 c_1)! ) = \sum_{i \geq 1} \frac{c_i}{(i + 1)!}.$$

Then $v(2\hat{\mathbb{Z}}) = [0, \frac{1}{2}]$, $v(1 + 2\hat{\mathbb{Z}}) = [\frac{1}{2}, 1]$, $v(1 + 6\hat{\mathbb{Z}}) = [\frac{1}{2}, \frac{2}{3}]$. 
Visualising profinite numbers

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One has

$$\#v^{-1}r = 2 \text{ for } r \in \mathbb{Q} \cap (0, 1),$$

$$\#v^{-1}r = 1 \text{ for all other } r \in [0, 1].$$

Examples:

$$v^{-1}\frac{1}{2} = \{-2, 1\}, \quad v^{-1}\frac{2}{3} = \{-5, 3\}, \quad v^{-1}1 = \{-1\}.$$
Graphs

For graphical purposes, we represent $a \in \hat{\mathbb{Z}}$ by $v(a) \in [0, 1]$.

We visualise a function $f : \hat{\mathbb{Z}} \to \hat{\mathbb{Z}}$ by representing its graph $\{(a, f(a)) : a \in \hat{\mathbb{Z}}\}$ in $[0, 1] \times [0, 1]$. 
Illustration by Willem Jan Palenstijn

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Four functions

In green: the graph of $a \mapsto a$.

In blue: the graph of $a \mapsto -a$.

In yellow: the graph of $a \mapsto a^{-1} - 1$ ($a \in \mathbb{Z}^*$).

In orange/red/brown: the graph of $a \mapsto F(a)$, the “$a$-th Fibonacci number”.
A more satisfactory definition is

\[ \hat{\mathbb{Z}} = \left\{ (a_n)_{n=1}^\infty \in \prod_{n=1}^\infty (\mathbb{Z}/n\mathbb{Z}) : n \mid m \Rightarrow a_m \equiv a_n \mod n \right\}. \]

This is a subring of \( \prod_{n=1}^\infty (\mathbb{Z}/n\mathbb{Z}) \).

Its unit group \( \hat{\mathbb{Z}}^* \) is a subgroup of \( \prod_{n=1}^\infty (\mathbb{Z}/n\mathbb{Z})^* \).
A formal definition

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\[ \hat{\mathbb{Z}} = \{(a_n)_{n=1}^\infty \in \prod_{n=1}^\infty (\mathbb{Z}/n\mathbb{Z}) : n|m \Rightarrow a_m \equiv a_n \mod n \}. \]

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Alternative definition: \( \hat{\mathbb{Z}} = \text{End}(\mathbb{Q}/\mathbb{Z}) \), the endomorphism ring of the abelian group \( \mathbb{Q}/\mathbb{Z} \). Then \( \hat{\mathbb{Z}}^* = \text{Aut}(\mathbb{Q}/\mathbb{Z}) \).
Basic facts

The ring \( \hat{\mathbb{Z}} \) is *uncountable*, it is *commutative*, and it has \( \mathbb{Z} \) as a subring. It has lots of zero-divisors.
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The ring $\hat{\mathbb{Z}}$ is *uncountable*, it is *commutative*, and it has $\mathbb{Z}$ as a subring. It has lots of zero-divisors.

For each $m \in \mathbb{Z}_{>0}$, there is a ring homomorphism

$$\hat{\mathbb{Z}} \to \mathbb{Z}/m\mathbb{Z}, \quad a = (a_n)_{n=1}^{\infty} \mapsto a_m,$$

which together with the group homomorphism $\hat{\mathbb{Z}} \to \hat{\mathbb{Z}}$, $a \mapsto ma$, fits into a short exact sequence

$$0 \to \hat{\mathbb{Z}} \xrightarrow{m} \hat{\mathbb{Z}} \to \mathbb{Z}/m\mathbb{Z} \to 0.$$
Profinite rationals

Write

\[ \hat{\mathbb{Q}} = \left\{ (a_n)_{n=1}^\infty \in \prod_{n=1}^\infty (\mathbb{Q}/n\mathbb{Z}) : n|m \Rightarrow a_m \equiv a_n \mod n\mathbb{Z} \right\}. \]

The additive group \( \hat{\mathbb{Q}} \) has exactly one ring multiplication extending the ring multiplication on \( \hat{\mathbb{Z}} \).
Profinite rationals

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\[ \hat{Q} = \left\{ (a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \left( \mathbb{Q}/n\mathbb{Z} \right) : n|m \Rightarrow a_m \equiv a_n \mod n\mathbb{Z} \right\}. \]

The additive group \( \hat{Q} \) has exactly one ring multiplication extending the ring multiplication on \( \hat{\mathbb{Z}} \).

It is a commutative ring, with \( \mathbb{Q} \) and \( \hat{\mathbb{Z}} \) as subrings, and

\[ \hat{Q} = Q + \hat{\mathbb{Z}} = Q \cdot \hat{\mathbb{Z}} \cong Q \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \]

(as rings).
If each $\mathbb{Z}/n\mathbb{Z}$ has the discrete topology and $\prod_{n=1}^{\infty}(\mathbb{Z}/n\mathbb{Z})$ the product topology, then $\hat{\mathbb{Z}}$ is closed in $\prod_{n=1}^{\infty}(\mathbb{Z}/n\mathbb{Z})$. 
Topology

If each $\mathbb{Z}/n\mathbb{Z}$ has the discrete topology and $\prod_{n=1}^{\infty}(\mathbb{Z}/n\mathbb{Z})$ the product topology, then $\hat{\mathbb{Z}}$ is closed in $\prod_{n=1}^{\infty}(\mathbb{Z}/n\mathbb{Z})$.

One can define the topology on $\hat{\mathbb{Z}}$ by the metric

$$d(x, y) = \frac{1}{\min\{k \in \mathbb{Z}_{>0} : x \not\equiv y \mod (k + 1)!\}}$$

$$= \frac{1}{\min\{k \in \mathbb{Z}_{>0} : c_k \neq d_k\}}$$

if $x = (\ldots c_3c_2c_1)!$, $y = (\ldots d_3d_2d_1)!$, $x \neq y$. 
More topology

*Fact:* \( \mathbb{Z} \) is a compact Hausdorff totally disconnected topological ring.

One can make the map \( v: \mathbb{Z} \to [0, 1] \) into a homeomorphism by “cutting” \([0, 1]\) at every \( r \in \mathbb{Q} \cap (0, 1) \).
More topology

*Fact:* $\hat{\mathbb{Z}}$ is a compact Hausdorff totally disconnected topological ring.

One can make the map $v : \hat{\mathbb{Z}} \to [0, 1]$ into a homeomorphism by “cutting” $[0, 1]$ at every $r \in \mathbb{Q} \cap (0, 1)$.

A neighborhood base of 0 in $\hat{\mathbb{Z}}$ is $\{m\hat{\mathbb{Z}} : m \in \mathbb{Z}_{>0}\}$.

With the same neighborhood base, $\hat{\mathbb{Q}}$ is also a topological ring. It is *locally* compact, Hausdorff, and totally disconnected.
Amusements for algebraists

We have $\hat{\mathbb{Z}} \subset A = \prod_{n=1}^{\infty} (\mathbb{Z}/n\mathbb{Z})$.

**Theorem.** One has $A/\hat{\mathbb{Z}} \cong A$ as additive topological groups.

**Proof** (Carlo Pagano): write down a surjective continuous group homomorphism $\epsilon: A \rightarrow A$ with $\ker \epsilon = \hat{\mathbb{Z}}$.
Amusements for algebraists

We have \( \hat{\mathbb{Z}} \subset A = \prod_{n=1}^{\infty} (\mathbb{Z}/n\mathbb{Z}) \).

**Theorem.** One has \( A/\hat{\mathbb{Z}} \cong A \) as additive topological groups.

**Proof** (Carlo Pagano): write down a surjective continuous group homomorphism \( \epsilon: A \to A \) with \( \ker \epsilon = \hat{\mathbb{Z}} \).

**Theorem.** One has \( A \cong A \times \hat{\mathbb{Z}} \) as groups but not as topological groups.

Here the axiom of choice comes in.
Profinite groups

In infinite Galois theory, the Galois groups that one encounters are *profinite groups*.

A profinite group is a topological group that is isomorphic to a closed subgroup of a product of finite discrete groups.

Equivalent definition: it is a compact Hausdorff totally disconnected topological group.

*Examples*: the additive group of $\hat{\mathbb{Z}}$ and its unit group $\hat{\mathbb{Z}}^*$ are profinite groups.
\( \hat{\mathbb{Z}} \) as the analogue of \( \mathbb{Z} \)

*Familiar fact.* For each group \( G \) and each \( \gamma \in G \) there is a unique group homomorphism \( \mathbb{Z} \rightarrow G \) with \( 1 \mapsto \gamma \), namely \( n \mapsto \gamma^n \).

*Analogue for \( \hat{\mathbb{Z}} \).* For each profinite group \( G \) and each \( \gamma \in G \) there is a unique group homomorphism \( \hat{\mathbb{Z}} \rightarrow G \) with \( 1 \mapsto \gamma \), and it is continuous. Notation: \( a \mapsto \gamma^a \).
Examples of infinite Galois groups

For a field $k$, denote by $\bar{k}$ an algebraic closure.

*Example 1:* with $p$ prime and $\mathbf{F}_p = \mathbb{Z}/p\mathbb{Z}$ one has

$$\hat{\mathbb{Z}} \cong \text{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p), \quad a \mapsto \text{Frob}^a,$$

where $\text{Frob}(\alpha) = \alpha^p$ for all $\alpha \in \bar{\mathbf{F}}_p$. 

*Example 2:* with $\mu = \{\text{roots of unity in } \bar{\mathbb{Q}}\} \cong \mathbb{Q}/\mathbb{Z}$ one has

$$\text{Gal}(\mathbb{Q}(\mu)/\mathbb{Q}) \cong \hat{\mathbb{Z}}^\ast$$

as topological groups.
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$$\mu = \{\text{roots of unity in } \bar{\mathbb{Q}}^*\} \cong \mathbb{Q}/\mathbb{Z}$$

one has

$$\text{Gal}(\mathbb{Q}(\mu)/\mathbb{Q}) \cong \text{Aut} \mu \cong \hat{\mathbb{Z}}^*$$

as topological groups.
Radical Galois groups

Example 3. For \( r \in \mathbb{Q} \), \( r \notin \{-1, 0, 1\} \), put

\[
\sqrt[n]{r} = \{ \alpha \in \bar{\mathbb{Q}} : \exists n \in \mathbb{Z}_{>0} : \alpha^n = r \}.
\]

**Theorem** (Abtien Javanpeykar). Let \( G \) be a profinite group. Then there exists \( r \in \mathbb{Q}\setminus\{-1, 0, 1\} \) with \( G \cong \text{Gal}(\mathbb{Q}(\sqrt[n]{r})/\mathbb{Q}) \) (as topological groups) if and only if there is a non-split exact sequence

\[
0 \to \hat{\mathbb{Z}} \xrightarrow{\iota} G \xrightarrow{\pi} \hat{\mathbb{Z}}^* \to 1
\]

of profinite groups such that

\[
\forall a \in \hat{\mathbb{Z}}, \gamma \in G : \gamma \cdot \iota(a) \cdot \gamma^{-1} = \iota(\pi(\gamma) \cdot a).
\]
Given $f_1, \ldots, f_k \in \mathbb{Z}[X_1, \ldots, X_n]$, one wants to solve the system $f_1(x) = \ldots = f_k(x) = 0$ in $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$.

**Theorem.** (a) There is a solution $x \in \mathbb{Z}^n \Rightarrow$ for each $m \in \mathbb{Z}_{>0}$ there is a solution modulo $m \iff$ there is a solution $x \in \hat{\mathbb{Z}}^n$.

(b) It is decidable whether a given system has a solution $x \in \hat{\mathbb{Z}}^n$. 

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$p$-adic numbers

Let $p$ be prime. The ring of $p$-adic integers is

$$\mathbb{Z}_p = \{(b_i)_{i=0}^\infty \in \prod_{i=0}^\infty (\mathbb{Z}/p^i\mathbb{Z}) : i \leq j \Rightarrow b_j \equiv b_i \text{ mod } p^i \}.$$ 

Just as $\hat{\mathbb{Z}}$, it is a compact Hausdorff totally disconnected topological ring.
\( p \)-adic numbers

Let \( p \) be prime. The \textit{ring of \( p \)-adic integers} is

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\mathbb{Z}_p = \{(b_i)_{i=0}^{\infty} \in \prod_{i=0}^{\infty} (\mathbb{Z}/p^i\mathbb{Z}) : i \leq j \Rightarrow b_j \equiv b_i \mod p^i \}.
\]

Just as \( \hat{\mathbb{Z}} \), it is a compact Hausdorff totally disconnected topological ring.

It is also a \textit{principal ideal domain}, with \( p\mathbb{Z}_p \) as its only non-zero prime ideal. Its field of fractions is written \( \mathbb{Q}_p \).

All ideals of \( \mathbb{Z}_p \) are \textit{closed}, and of the form \( p^h\mathbb{Z}_p \) with \( h \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \), where \( p^\infty\mathbb{Z}_p = \{0\} \).
The Chinese remainder theorem

For \( n = \prod_{p \text{ prime}} p^{i(p)} \) one has

\[
\mathbb{Z}/n\mathbb{Z} \cong \prod_{p \text{ prime}} (\mathbb{Z}/p^{i(p)}\mathbb{Z}) \quad \text{(as rings)}.
\]

In the limit:

\[
\hat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p \quad \text{(as topological rings)}.
\]
The Chinese remainder theorem

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$$\mathbb{Z}/n\mathbb{Z} \cong \prod_{p \text{ prime}} (\mathbb{Z}/p^{i(p)}\mathbb{Z}) \quad (\text{as rings}).$$

In the limit:

$$\hat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p \quad (\text{as topological rings}).$$

For each $p$, the projection map $\hat{\mathbb{Z}} \to \mathbb{Z}_p$ induces a ring homomorphism $\pi_p : \hat{\mathbb{Q}} \to \mathbb{Q}_p$. 
Profinite number theory

The isomorphism $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ reduces most questions that one may ask about $\hat{\mathbb{Z}}$ to similar questions about the much better behaved rings $\mathbb{Z}_p$.

Profinite number theory studies the exceptions. Many of these are caused by the set $\mathcal{P}$ of primes being infinite.
Ideals of $\hat{\mathbb{Z}}$

For an ideal $a \subset \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, one has:

\[ a \text{ is closed} \iff a \text{ is finitely generated} \iff a \text{ is principal} \iff a = \prod_p a_p \text{ where each } a_p \subset \mathbb{Z}_p \text{ an ideal}. \]

The set of closed ideals of $\hat{\mathbb{Z}}$ is in bijection with the set \( \{ \prod_p p^{h(p)} : h(p) \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \} \) of Steinitz numbers.

Most ideals of $\hat{\mathbb{Z}}$ are not closed.
The spectrum and ultrafilters

The *spectrum* $\text{Spec } R$ of a commutative ring $R$ is its set of prime ideals. *Example*: $\text{Spec } \mathbb{Z}_p = \{\{0\}, p\mathbb{Z}_p\}$. 

With each $p \in \text{Spec } \hat{\mathbb{Z}}$ one associates the ultrafilter $\Upsilon(p) = \{S \subseteq \mathbb{P} : e_S \in p\}$ on the set $\mathbb{P}$ of primes, where $e_S \in \prod_{p \in \mathbb{P}} \mathbb{Z}_p = \hat{\mathbb{Z}}$ has coordinate 0 at $p \in S$ and 1 at $p \not\in S$. Then $p$ is closed if and only if $\Upsilon(p)$ is principal, and $\Upsilon(p) = \Upsilon(q)$ $\iff$ $p \subseteq q$ or $q \subseteq p$. 

Profinite number theory Hendrik Lenstra
The spectrum and ultrafilters

The *spectrum* Spec $R$ of a commutative ring $R$ is its set of prime ideals. *Example:* $\text{Spec } \mathbb{Z}_p = \{\{0\}, p\mathbb{Z}_p\}$.

With each $p \in \text{Spec } \hat{\mathbb{Z}}$ one associates the *ultrafilter*

$$\Upsilon(p) = \{S \subset \mathcal{P} : e_S \in p\}$$

on the set $\mathcal{P}$ of primes, where $e_S \in \prod_{p \in \mathcal{P}} \mathbb{Z}_p = \hat{\mathbb{Z}}$ has coordinate 0 at $p \in S$ and 1 at $p \notin S$.

Then $p$ is closed if and only if $\Upsilon(p)$ is principal, and

$$\Upsilon(p) = \Upsilon(q) \iff p \subset q \text{ or } q \subset p.$$
The logarithm

\[ u \in \mathbb{R}_{>0} \Rightarrow \log u = \left( \frac{d}{dx} u^x \right)_{x=0} = \lim_{\epsilon \to 0} \frac{u^\epsilon - 1}{\epsilon}. \]
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Analogously, define \( \log : \hat{\mathbb{Z}}^* \to \hat{\mathbb{Z}} \) by

\[ \log u = \lim_{n \to \infty} \frac{u^n! - 1}{n!} . \]

This is a well-defined continuous group homomorphism.
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This is a well-defined continuous group homomorphism.

Its kernel is \( \hat{\mathbb{Z}}_{\text{tor}}^* \), which is the closure of the set of elements of finite order in \( \hat{\mathbb{Z}}^* \).

Its image is \( 2J = \{2x : x \in J\} \), where \( J = \bigcap_p p\hat{\mathbb{Z}} \) is the Jacobson radical of \( \hat{\mathbb{Z}} \).
Structure of $\hat{\mathbb{Z}}^*$

The logarithm fits in a commutative diagram

$$
\begin{array}{cccccc}
1 & \rightarrow & \hat{\mathbb{Z}}^*_{\text{tor}} & \rightarrow & \hat{\mathbb{Z}}^* & \stackrel{\text{log}}{\rightarrow} & 2J & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \leftarrow & (\hat{\mathbb{Z}}/2J)^* & \leftarrow & \hat{\mathbb{Z}}^* & \leftarrow & 1 + 2J & \leftarrow & 1
\end{array}
$$

of profinite groups, where the other horizontal maps are the natural ones, the rows are exact, and the vertical maps are isomorphisms.

Corollary: $\hat{\mathbb{Z}}^* \cong (\hat{\mathbb{Z}}/2J)^* \times 2J$ (as topological groups).
More on $\hat{\mathbb{Z}}^*$

Less canonically, with $A = \prod_{n \geq 1} (\mathbb{Z}/n\mathbb{Z})$:

$$2J \cong \hat{\mathbb{Z}},$$

$$(\hat{\mathbb{Z}}/2J)^* \cong (\mathbb{Z}/2\mathbb{Z}) \times \prod_p (\mathbb{Z}/(p - 1)\mathbb{Z}) \cong A,$$

$$\hat{\mathbb{Z}}^* \cong A \times \hat{\mathbb{Z}},$$

as topological groups, and

$$\hat{\mathbb{Z}}^* \cong A$$

as groups.
Power series expansions

The inverse isomorphisms

\[ \log: \ 1 + 2J \rightsquigarrow 2J \]
\[ \exp: \ 2J \rightsquigarrow 1 + 2J \]

are given by power series expansions

\[ \log(1 - x) = - \sum_{n=1}^{\infty} \frac{x^n}{n}, \quad \exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

that converge for all \( x \in 2J \).

The logarithm is analytic on all of \( \hat{\mathbb{Z}}^* \) in a weaker sense.
Analyticity

Let $x_0 \in D \subset \hat{\mathbb{Q}}$. We call $f : D \to \hat{\mathbb{Q}}$ analytic in $x_0$ if there is a sequence $(a_n)_{n=0}^\infty \in \hat{\mathbb{Q}}^\infty$ such that one has

$$f(x) = \sum_{n=0}^\infty a_n \cdot (x - x_0)^n$$

in the sense that for each prime $p$ there is a neighborhood $U$ of $x_0$ in $D$ such that for all $x \in U$ the equality

$$\pi_p(f(x)) = \sum_{n=0}^\infty \pi_p(a_n) \cdot (\pi_p(x) - \pi_p(x_0))^n$$

is valid in the topological field $\mathbb{Q}_p$. 
Examples of analytic functions

The map $\log: \hat{\mathbb{Z}}^* \to \hat{\mathbb{Z}} \subset \hat{\mathbb{Q}}$ is analytic in each $x_0 \in \hat{\mathbb{Z}}^*$, with expansion

$$\log x = \log x_0 - \sum_{n=1}^{\infty} \frac{(x_0 - x)^n}{n \cdot x_0^n}.$$
Examples of analytic functions

The map \( \hat{\mathbb{Z}}^* \rightarrow \hat{\mathbb{Z}} \subset \hat{\mathbb{Q}} \) is analytic in each \( x_0 \in \hat{\mathbb{Z}}^* \), with expansion

\[
\log x = \log x_0 - \sum_{n=1}^{\infty} \frac{(x_0 - x)^n}{n \cdot x_0^n}.
\]

For each \( u \in \hat{\mathbb{Z}}^* \), the map

\[
\hat{\mathbb{Z}} \rightarrow \hat{\mathbb{Z}}^* \subset \hat{\mathbb{Q}}, \quad x \mapsto u^x
\]

is analytic in each \( x_0 \in \hat{\mathbb{Z}} \), with expansion

\[
u^x = \sum_{n=0}^{\infty} \frac{(\log u)^n \cdot u^x_0 \cdot (x - x_0)^n}{n!}.
\]
A Fibonacci example

Define $F : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ by

$$
F(0) = 0, \quad F(1) = 1, \quad F(n + 2) = F(n + 1) + F(n).
$$

**Theorem.** The function $F$ has a unique continuous extension $\hat{\mathbb{Z}} \rightarrow \hat{\mathbb{Z}}$, and it is analytic in each $x_0 \in \hat{\mathbb{Z}}$.

Notation: $F$. 

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A Fibonacci example

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Notation: $F$.

For $n \in \mathbb{Z}$, one has
\[
F(n) = n \Leftrightarrow n \in \{0, 1, 5\}.
\]
Up to eleven

One has $\#\{x \in \hat{\mathbb{Z}} : F(x) = x\} = 11$.

The only even fixed point of $F$ is 0, and for each $a \in \{1, 5\}$, $b \in \{-5, -1, 0, 1, 5\}$ there is a unique fixed point $z_{a,b}$ with

$$z_{a,b} \equiv a \mod \bigcap_{n=0}^{\infty} 6^n \hat{\mathbb{Z}}, \quad z_{a,b} \equiv b \mod \bigcap_{n=0}^{\infty} 5^n \hat{\mathbb{Z}}.$$

Examples: $z_{1,1} = 1$, $z_{5,5} = 5$. 
Illustration by Willem Jan Palenstijn

Profinite number theory
Hendrik Lenstra
Graphing the fixed points

The graph of \(a \mapsto F(a)\) is shown in orange/red/brown.

Intersecting the graph with the diagonal one obtains the fixed points 0 and \(z_{a,b}\), for \(a = 1, 5, b = -5, -1, 0, 1, 5\).
Graphing the fixed points

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*Surprise*: one has $z_{5,-5}^2 - 25 = \sum_{i=1}^{\infty} c_i i!$ with $c_i = 0$ for $i \leq 200$ and $c_{201} \neq 0$. 
Larger cycles

I believe:

$$\#\{x \in \hat{\mathbb{Z}} : F(F(x)) = x\} = 21,$$
$$\#\{x \in \hat{\mathbb{Z}} : F^n(x) = x\} < \infty \quad \text{for each } n \in \mathbb{Z}_{>0}.$$

**Question:** does $F$ have cycles of length greater than 2?
Other linear recurrences

If $E : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$, $t \in \mathbb{Z}_{> 0}$, $d_0$, $\ldots$, $d_{t-1} \in \mathbb{Z}$ satisfy

$$\forall n \in \mathbb{Z}_{\geq 0} : E(n + t) = \sum_{i=0}^{t-1} d_i \cdot E(n + i),$$

d_0 \in \{1, -1\},

then $E$ has a unique continuous extension $\hat{\mathbb{Z}} \to \hat{\mathbb{Z}}$. It is analytic in each $x_0 \in \hat{\mathbb{Z}}$. 
Suppose also \( X^t - \sum_{i=0}^{t-1} d_i X^i = \prod_{i=1}^{t} (X - \alpha_i) \), where \( \alpha_1, \ldots, \alpha_t \in \mathbb{Q}(\sqrt{Q}) \), \( \alpha_j^{24} \neq \alpha_k^{24} \) (1 \( \leq j < k \leq t \)).

**Tentative theorem.** If \( n \in \mathbb{Z}_{>0} \) is such that the set

\[
S_n = \{ x \in \hat{\mathbb{Z}} : E^n(x) = x \}
\]

is infinite, then \( S_n \cap \mathbb{Z}_{\geq 0} \) contains an infinite arithmetic progression.
Finite cycles

Suppose also $X^t - \sum_{i=0}^{t-1} d_i X^i = \prod_{i=1}^{t} (X - \alpha_i)$, where

$$\alpha_1, \ldots, \alpha_t \in \mathbb{Q}(\sqrt{Q}),$$

$$\alpha_j^{24} \neq \alpha_k^{24} \quad (1 \leq j < k \leq t).$$

**Tentative theorem.** If $n \in \mathbb{Z}_{>0}$ is such that the set

$$S_n = \{ x \in \hat{\mathbb{Z}} : E^n(x) = x \}$$

is infinite, then $S_n \cap \mathbb{Z}_{\geq 0}$ contains an infinite arithmetic progression.

This would imply that $\{ x \in \hat{\mathbb{Z}} : F^n(x) = x \}$ is finite for each $n \in \mathbb{Z}_{>0}$. 