Diagonal cycles and Euler systems for real quadratic fields

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An ongoing project with

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The Birch and Swinnerton-Dyer conjecture

One of the major outstanding issues in the Birch and Swinnerton-Dyer conjecture is the (explicit) construction of rational points on elliptic curves.

There are *very few* strategies for producing such rational points:

1. Heegner points (CM points on modular elliptic curves). Birch, Gross-Zagier-Zhang, Kolyvagin...
2. Higher-dimensional algebraic cycles can sometimes be used to construct "interesting" rational points on elliptic curves, as described in Victor's lecture.

Summary of Victor's Lecture

Cycle classes in the triple product of modular curves lead to rational points on elliptic curves.

These points make it possible to relate:

1. Certain extension classes (of mixed motives) arising in the pro-unipotent fundamental groups of modular curves;
2. Special values of L-functions of modular forms.

This fits into a general philosophy (Deligne, Wojtkowiak, ...) relating $\pi_1^{\mathrm{unip}}(X)$ to values of L-functions.

What about BSD?

Question: Do these constructions yield anything "genuinely new" about the Birch and Swinnerton-Dyer conjecture for elliptic curves over $\mathbb{Q}$?

BSD Conjecture: $r_{\mathrm{an}}(E/\mathbb{Q}) = r(E/\mathbb{Q})$, where

$$r_{\mathrm{an}}(E/\mathbb{Q}) := \mathrm{ord}_{s=1} L(E/\mathbb{Q}, s), \qquad r(E/\mathbb{Q}) = \mathrm{rank}(E(\mathbb{Q})).$$

$r_{\mathrm{an}}(E/\mathbb{Q}) \leq 1$: everything is known.

$r_{\mathrm{an}}(E/\mathbb{Q}) > 1$: we haven't the slightest idea.

A "equivariant" BSD conjecture

L-functions carry a lot of information about the structure of $E(\bar{\mathbb{Q}})$.

Consider a continuous Artin representation

$$\rho : \mathrm{Gal}(K_\rho/\mathbb{Q}) \longrightarrow \mathsf{GL}_n(\mathbb{C}).$$

$$r_{\mathrm{an}}(E, \rho) := \mathrm{ord}_{s=1} L(E, \rho, s),$$

$$r(E, \rho) := \dim_{\mathbb{C}} \hom(V_\rho, E(K_\rho) \otimes \mathbb{C}).$$

Conjecture ("Equivariant" BSD)

For all Artin representations ρ, $r_{\mathrm{an}}(E, \rho) = r(E, \rho)$.

What is known?

The following cases of the conjecture have been established:

1. ρ is one-dimensional (i.e., corresponds to a Dirichlet character), and $r_{\mathrm{an}}(E, \rho) = 0$. (Kato, 1991).
2. $\rho = \mathrm{Ind}_K^{\mathbb{Q}} \chi$, where χ=dihedral, K=quadratic imaginary field, $r_{\mathrm{an}}(E, \rho) = 1$. (Kolyvagin, Gross-Zagier, Zhang, ..., 1989).
3. Similar setting, $r_{\mathrm{an}}(E, \rho) = 0$. (Bertolini-D, Rotger, Vigni, Nekovar,... ,1996).

Artin Representations

We will be primarily interested in *odd* Artin representations

$$\rho : \mathrm{Gal}(K_\rho/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{C}).$$

The cases that can arise are:

1. $\rho = \mathrm{Ind}_K^{\mathbb{Q}} \chi$, where K = imaginary quadratic field;
2. $\rho = \mathrm{Ind}_F^{\mathbb{Q}} \chi$, where F = real quadratic field, and χ has signature $(+, -)$.
3. The projective image of ρ is A_4, S_4 or A_5.

The BSD theorem

$$E = \text{elliptic curve over } \mathbb{Q};$$

$$\rho_1, \rho_2 = \text{ odd 2-dimensional representations of } G_{\mathbb{Q}},$$

$$\det(\rho_1)\det(\rho_2) = 1.$$

The following theorem is the the primary goal of the current project with V. Rotger.

Theorem (Rotger, D: still in progress, and far from complete!)

Assume that there exists $\sigma \in G_{\mathbb{Q}}$ for which $\rho_1 \otimes \rho_2(\sigma)$ has distinct eigenvalues. If $L(E \otimes \rho_1 \otimes \rho_2, 1) \neq 0$, then

$$\hom(V_{\rho_1} \otimes V_{\rho_2}, E(K_{\rho_1} K_{\rho_2}) \otimes \mathbb{C}) = 0.$$

Modularity

The objects E, ρ_1, and ρ_2 are all known to be modular!

As usual, this plays a key role.

Theorem (Hecke, Langlands-Tunnell, Wiles, Taylor, Khare,. . .)

There exist modular forms f of weight two, and g and h of weight one, such that

$$L(f, s) = L(E, s), \qquad L(g, s) = L(\rho_1, s), \quad L(h, s) = L(\rho_2, s).$$

Strategy of the proof

The strategy for the proof of our sought-for Theorem rests on the following key ingredients.

1. Galois cohomology classes

$$\kappa(f, g', h') \in H^1(\mathbb{Q}, V_f \otimes V_{g'} \otimes V_{h'})$$

   attached to a *triple* (f, g', h') of modular forms of weights ≥ 2. They are obtained from the image of diagonal cycles on triple products of Kuga-Sato varieties under p-adic étale Abel-Jacobi maps.

2. p-adic deformations of these classes, attached to Hida families $\underline{f}$, $\underline{g}$ and $\underline{h}$ interpolating f, g and h.
3. Various relations between these classes and L-functions (both complex and p-adic) attached to $\underline{f} \otimes \underline{g} \otimes \underline{h}$.

Triples of eigenforms

Definition

A triple of eigenforms

$$f \in S_k(\Gamma_0(N_f), \varepsilon_f), \quad g \in S_\ell(\Gamma_0(N_g), \varepsilon_g), \quad h \in S_m(\Gamma_0(N_h), \varepsilon_h)$$

is said to be *critical* if

1. Their weights are *balanced*:

$$k < \ell + m, \quad \ell < k + m, \quad m < k + \ell.$$

2. $\varepsilon_f \varepsilon_g \varepsilon_h = 1$; in particular, $k + \ell + m$ is even.

Diagonal cycles on triple products of Kuga-Sato varieties.

$$k = r_1 + 2, \quad \ell = r_2 + 2, \quad m = r_3 + 2, \quad r = \frac{r_1 + r_2 + r_3}{2}.$$

$\mathcal{E}^r(N) = r$-fold Kuga-Sato variety over $X_1(N)$; $\dim = r + 1$.

$$V = \mathcal{E}^{r_1}(N_f) \times \mathcal{E}^{r_2}(N_g) \times \mathcal{E}^{r_3}(N_h), \quad \dim V = 2r + 3.$$

Generalised Gross-Kudla-Schoen cycle: there is an *essentially unique* interesting way of embedding $\mathcal{E}^r(N_f N_g N_h)$ as a null-homologous cycle in V.

Cf. Rotger, D. Notes for the AWS, Chapter 7.

$$\Delta = \mathcal{E}^r \subset V, \quad \Delta \in CH^{r+2}(V).$$

Diagonal cycles and L-series

The height of the (f, g, h)-isotypic component $\Delta^{f,g,h}$ of the Gross-Kudla-Schoen cycle Δ should be related to the central critical derivative

$$L'(f \otimes g \otimes h, r+2).$$

Work of Yuan-Zhang-Zhang represents substantial progress in this direction, when $r_1 = r_2 = r_3 = 0$.

Our goal will be instead: to describe other relationships between $\Delta^{f,g,h}$ and p-adic L-series attached to (f, g, h), in view of obtaining the arithmetic applications described above.

Complex Abel-Jacobi maps

The cycle Δ is null-homologous:

$$\mathrm{cl}(\Delta) = 0 \text{ in } H^{2r+4}(V(\mathbb{C}), \mathbb{Q}).$$

Our formula of "Gross-Kudla-Zhang type" will not involve heights, but rather p-adic analogues of the *complex Abel-Jacobi map* of Griffiths and Weil:

$$\begin{aligned} \mathrm{AJ} : \mathrm{CH}^{r+2}(V)_0 \quad &\longrightarrow \quad \frac{H_{\mathrm{dR}}^{2r+3}(V/\mathbb{C})}{\mathrm{Fil}^{r+2} H_{\mathrm{dR}}^{2r+3}(V/\mathbb{C}) + H_B^{2r+3}(V(\mathbb{C}), \mathbb{Z})} \\ &= \frac{\mathrm{Fil}^{r+2} H_{\mathrm{dR}}^{2r+3}(V/\mathbb{C})^\vee}{H_{2r+3}(V(\mathbb{C}), \mathbb{Z})}. \end{aligned}$$

$$\mathrm{AJ}(\Delta)(\omega) = \int_{\partial^{-1}\Delta} \omega.$$

p-adic étale Abel-Jacobi maps

$$\begin{tikzcd}
\mathrm{CH}^{r+2}(V/\mathbb{Q})_0 \arrow[r, "\mathrm{AJ}_{\mathrm{et}}"] \arrow[d] \arrow[ddr, dotted] & H^1_f(\mathbb{Q}, H^{2r+3}_{\mathrm{et}}(\bar{V}, \mathbb{Q}_p)(r+2)) \arrow[d] \\
\mathrm{CH}^{r+2}(V/\mathbb{Q}_p)_0 \arrow[r, "\mathrm{AJ}_{\mathrm{et}}"'] & H^1_f(\mathbb{Q}_p, H^{2r+3}_{\mathrm{et}}(\bar{V}, \mathbb{Q}_p)(r+2)) \arrow[d, equal] \\
& \mathrm{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(V/\mathbb{Q}_p)^\vee
\end{tikzcd}$$

The dotted arrow is called the *p-adic Abel-Jacobi map* and denoted AJ_p.

Goal: Relate $\mathrm{AJ}_p(\Delta)$ to *certain* Rankin triple product p-adic L-functions, à la Gross-Kudla-Zhang.

Hida families

Let p be any prime, and replace f, g and h by their p-stabilisations, which are both *ordinary* (eigenvectors for U_p with eigenvalue a p-adic unit).

Theorem (Hida)

There exist p-adic families

$$\underline{f}(k) = \sum a_n(k)q^n, \qquad \underline{g}(\ell) = \sum b_n(\ell)q^n, \quad \underline{h}(m) = \sum c_n(m)q^n,$$

($k, \ell, m \in \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$) of modular forms satisfying $\underline{f}(2) = f$, $\underline{g}(1) = g$ and $\underline{h}(1) = h$.

For $k, \ell, m \in \mathbb{Z}^{\geq 2}$, the specialisations

$$f_k := \underline{f}(k), \qquad g_\ell := \underline{g}(\ell), \qquad h_m := \underline{h}(m)$$

are classical eigenforms of weights k, ℓ and m.

Triple product p-adic Rankin L-functions

They interpolate the *central critical values*

$$\frac{L(\underline{f}(k) \otimes \underline{g}(\ell) \otimes \underline{h}(m), \frac{k+\ell+m-2}{2})}{\Omega(k,\ell,m)} \in \bar{\mathbb{Q}}.$$

Four *distinct* regions of interpolation:

1. $\Sigma_f = \{(k,\ell,m) : k \geq \ell + m\}$. $\Omega(k,\ell,m) = *\langle f_k, f_k \rangle^2$.
2. $\Sigma_g = \{(k,\ell,m) : \ell \geq k + m\}$. $\Omega(k,\ell,m) = *\langle g_\ell, g_\ell \rangle^2$.
3. $\Sigma_h = \{(k,\ell,m) : m \geq k + \ell\}$. $\Omega(k,\ell,m) = *\langle h_m, h_m \rangle^2$.
4. $\Sigma_{\text{bal}} = (\mathbb{Z}^{\geq 2})^3 - \Sigma_f - \Sigma_g - \Sigma_h$.
   $\Omega(k,\ell,m) = *\langle f_k, f_k \rangle^2 \langle g_\ell, g_\ell \rangle^2 \langle h_m, h_m \rangle^2$.

Resulting p-adic L-functions: $L_p^f(\underline{f} \otimes \underline{g} \otimes \underline{h}, k, \ell, m)$, $L_p^g(\underline{f} \otimes \underline{g} \otimes \underline{h}, k, \ell, m)$, and $L_p^h(\underline{f} \otimes \underline{g} \otimes \underline{h}, k, \ell, m)$ respectively.

More notations

$$\omega_f = (2\pi i)^{r_1+1} f(\tau) dw_1 \cdots dw_{r_1} d\tau \in \mathrm{Fil}^{r_1+1} H^{r_1+1}_{\mathrm{dR}}(\mathcal{E}^{r_1}).$$

$\eta_f \in H^{r_1+1}_{\mathrm{dR}}(\mathcal{E}^{r_1}/\bar{\mathbb{Q}}_p) =$ representative of the f-isotypic part on which Frobenius acts via $\alpha_p(f)$, normalised so that

$$\langle \omega_f, \eta_f \rangle = 1.$$

Lemma

If (k, ℓ, m) is balanced, then the (f_k, g_ℓ, h_m)-isotypic part of the $\bar{\mathbb{Q}}_p$ vector space $\mathrm{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(V/\bar{\mathbb{Q}}_p)$ is generated by the classes of

$$\omega_{f_k} \otimes \omega_{g_\ell} \otimes \omega_{h_m}, \quad \eta_{f_k} \otimes \omega_{g_\ell} \otimes \omega_{h_m}, \quad \omega_{f_k} \otimes \eta_{g_\ell} \otimes \omega_{h_m}, \quad \omega_{f_k} \otimes \omega_{g_\ell} \otimes \eta_{h_m}.$$

A p-adic Gross-Kudla formula

Assume that $\mathrm{sign}(L(f_k \otimes g_\ell \otimes h_m, s)) = -1$ for all $(k, \ell, m) \in \Sigma_{\mathrm{bal}}$. (For example, f, g and h are of the same level.)

Theorem (Rotger-Sols-D; in progress)

For all $(k, \ell, m) \in \Sigma_{\mathrm{bal}}$,

$$L_p^f(\underline{f} \otimes \underline{g} \otimes \underline{h}, k, \ell, m) = * \times \mathrm{AJ}_p(\Delta)(\eta_f \wedge \omega_g \wedge \omega_h),$$

and likewise for L_p^g *and* L_p^h.

Conclusion: The Abel-Jacobi image of Δ encodes the special values of the *three distinct* p-adic L-functions.

From cycles to cohomology classes

We can use the cycles $\Delta_{k,\ell,m}$ to construct global classes

$$\mathrm{AJ}_{\mathrm{et}}(\Delta_{k,\ell,m}) \in H^1(\mathbb{Q}, H^{2r+3}_{\mathrm{et}}(V_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(r+2)).$$

Künneth:

$$\begin{aligned} H^{2r+3}_{\mathrm{et}}(V_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(r+2) &\longrightarrow \bigotimes_{j=1}^{3} H^{r_j+1}_{\mathrm{et}}(\mathcal{E}^{r_j}_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(r+2) \\ &\longrightarrow V_{f_k} \otimes V_{g_\ell} \otimes V_{h_m}(r+2). \end{aligned}$$

By projecting $\mathrm{AJ}_{\mathrm{et}}(\Delta)$ we obtain a cohomology class

$$\kappa(f_k, g_\ell, h_m) \in H^1(\mathbb{Q}, V_{f_k} \otimes V_{g_\ell} \otimes V_{h_m}(r+2)),$$

for each $(k, \ell, m) \in \Sigma_{\mathrm{bal}}$.

The Birch-Swinnerton-Dyer class

We *really* want to construct a class in

$$H^1(\mathbb{Q}, V_f \otimes V_g \otimes V_h(1))$$

attached (formally) to the triple

$$(k, \ell, m) = (2, 1, 1) \in \Sigma_f.$$

Natural approach: interpolate the classes $\kappa(f_k, g_\ell, h_m)$ p-adically to extend their definition from Σ_{bal} to Σ_f.

The theme of p-adic variation

Slogan: The natural p-adic invariants attached to (classical) modular forms varying in p-adic families should also vary in p-adic families.

Example: The Serre-Deligne representation V_{g_ℓ} of $G_{\mathbb{Q}}$ attached to the classical eigenforms $\underline{g}(\ell)$ with $\ell \geq 2$.

Theorem

There exist Λ-adic representations $\underline{V}_g$ of $G_{\mathbb{Q}}$ satisfying

$$\underline{V}_g \otimes_{ev_\ell} \bar{\mathbb{Q}}_p = V_{g_\ell}\left(\frac{\ell-1}{2}\right), \qquad \text{for almost all } \ell \in \mathbb{Z}^{\geq 2} \cap U_g.$$

p-adic interpolation of diagonal cycle classes

For each $\ell \in \mathbb{Z}^{>1}$, the triple $(2, \ell, \ell)$ is balanced, so we can consider the cohomology classes

$$\kappa(f, g_\ell, h_\ell) \in H^1(\mathbb{Q}, V_f \otimes V_{g_\ell} \otimes V_{h_\ell}(\ell)).$$

$$\mathrm{ev}_{\ell,\ell} : \underline{V}_g \otimes \underline{V}_h \longrightarrow V_{g_\ell} \otimes V_{h_\ell}(\ell - 1).$$

Conjecture

There exists a "big" cohomology class

$$\underline{\kappa} \in H^1(\mathbb{Q}, \underline{V}_f \otimes \underline{V}_g \otimes \underline{V}_h(1))$$

such that

$$\underline{\kappa}(2, \ell, \ell) := \mathrm{ev}_{2,\ell,\ell}(\underline{\kappa}) = \kappa(f, g_\ell, h_\ell)$$

for almost all $\ell \in \mathbb{Z}^{\geq 2} \cap U_g \cap U_h$ *(note:* $(2, \ell, \ell) \in \Sigma_{\mathrm{bal}}$*).*

p-adic interpolation of cohomology classes

Similar interpolation results have been obtained and exploited in other contexts:

1. Kato: p-adic interpolation of classes arising from Beilinson elements in $H^1(\mathbb{Q}, V_p(f)(2))$. Their weight k specialisations encode higher weight Beilinson elements (A. Scholl, unpublished.)
2. Ben Howard: p-adic interpolation of classes arising from Heegner points. Their higher weight specialisations encode the images of higher weight Heegner cycles under p-adic Abel-Jacobi maps (Francesc Castella, in progress).

The BSD class

Assuming the construction of $\underline{\kappa}$, consider the specialisation

$$\begin{aligned} \underline{\kappa}(2,1,1) &\in H^1(\mathbb{Q}, V_f \otimes V_g \otimes V_h(1)) \\ &= H^1(\mathbb{Q}, V_p(E) \otimes \rho_1 \otimes \rho_2). \end{aligned}$$

The triple $(2,1,1) \notin \Sigma_{\mathrm{bal}}$, and therefore $\underline{\kappa}(2,1,1)$ lies *outside* the range of "geometric interpolation" defining the family $\underline{\kappa}$.

In particular, the restriction

$$\underline{\kappa}(2,1,1)_p \in H^1(\mathbb{Q}_p, V_p(E) \otimes \rho_1 \otimes \rho_2)$$

need not be cristalline.

The dual exponential map

p-adic exponential map:

$$\exp : \Omega^1(E/\mathbb{Q}_p)^\vee \longrightarrow E(\mathbb{Q}_p) \otimes \mathbb{Q}_p.$$

The dual map (exploiting Tate local duality):

$$\exp^* : \frac{H^1(\mathbb{Q}_p, V_p(E))}{H^1_f(\mathbb{Q}_p, V_p(E))} \longrightarrow \Omega^1(E/\mathbb{Q}_p).$$

Analogous map for $V_p(E) \otimes \rho_1 \otimes \rho_2$:

$$\exp^* : \frac{H^1(\mathbb{Q}_p, V_p(E) \otimes \rho_1 \otimes \rho_2)}{H^1_f(\mathbb{Q}_p, V_p(E) \otimes \rho_1 \otimes \rho_2)} \longrightarrow \Omega^1(E/\mathbb{Q}_p) \otimes \rho_1 \otimes \rho_2.$$

Question: Relate $\exp^*(\underline{\kappa}(2,1,1)) \in \Omega^1(E/\mathbb{Q}_p) \otimes \rho_1 \otimes \rho_2$ to L-functions.

A reciprocity law

Conjecture (Rotger, D)

The image of the class $\underline{\kappa}(2,1,1)$ *under* $\exp^*$ *has the following properties:*

1. *It belongs to* $\Omega^1(E/\mathbb{Q}_p) \otimes (\rho_1 \otimes \rho_2)^{\mathrm{frob}=\alpha_p(g)\alpha_p(h)}$;
2. *It is non-zero if and only if* $L(E \otimes \rho_1 \otimes \rho_2, 1) \neq 0$.

Heuristic, hand-waving argument for 2:

$$\begin{aligned}
\langle \exp^*(\underline{\kappa}(2,1,1)), \eta_f \omega_g \omega_h \rangle \quad & \rightsquigarrow \quad \lim_{(\ell,m)\to(1,1)} \mathrm{AJ}(\Delta)(\eta_f \otimes \omega_{g_\ell} \otimes \omega_{h_m}) \\
& \rightsquigarrow \quad \lim_{(\ell,m)\longrightarrow(1,1)} L_p^f(\underline{f} \otimes \underline{g} \otimes \underline{h}, 2, \ell, m) \\
& = \quad L_p^f(\underline{f} \otimes \underline{g} \otimes \underline{h}, 2, 1, 1) \\
& \rightsquigarrow \quad L(f \otimes g \otimes h, 1) \quad (2,1,1) \in \Sigma_f \ldots
\end{aligned}$$

Proof of the main theorem

Injection

$$\begin{aligned} \hom(\rho_1 \otimes \rho_2, E(\bar{\mathbb{Q}}) \otimes L) &\longrightarrow \hom(\rho_1 \otimes \rho_2, E(\bar{\mathbb{Q}}_p) \otimes L) \\ &= H^1_f(\mathbb{Q}_p, V_p(E) \otimes \rho_1 \otimes \rho_2) \end{aligned}$$

Exact sequence arising from local and global duality:

$$\begin{aligned} 0 &\longrightarrow \hom(\rho_1 \otimes \rho_2, E(\bar{\mathbb{Q}}) \otimes L) \longrightarrow H^1_f(\mathbb{Q}_p, V_p(E) \otimes \rho_1 \otimes \rho_2) \\ &\longrightarrow \left(\frac{H^1(\mathbb{Q}, V_p(E) \otimes \rho_1 \otimes \rho_2)}{H^1_f(\mathbb{Q}, V_p(E) \otimes \rho_1 \otimes \rho_2)} \right)^\vee . \end{aligned}$$

The parallel with Kato's method

This strategy is merely an adaptation of a method of Kato, in which families of Eisenstein series are replaced by families of cusp forms.

Kato	Rotger-D
(f, E_ℓ, F_m)	(f, g_ℓ, h_m)
Beilinson elements	Diagonal cycles
$L(f, j), j \geq 2$	$L(f \otimes g_\ell \otimes h_\ell, \ell)$
$\Downarrow$	$\Downarrow$
$L(f, 1)$	$L(f \otimes \rho_1 \otimes \rho_2, 1)$

Application to elliptic curves and real quadratic fields

Corollary

Let χ be a ring class character of a real quadratic field F. Then

$$L(E/F, \chi, 1) \neq 0 \implies (E(H) \otimes \mathbb{C})^{\chi} = 0.$$

Proof.

Find a character α of signature $(+,-)$ for which $L(E/F, \chi\alpha/\alpha', 1) \neq 0$.

$$\chi_1 = \chi\alpha, \quad \chi_2 = \alpha^{-1}, \qquad \rho_1 = \mathrm{Ind}_F^{\mathbb{Q}} \chi_1, \quad \rho_2 = \mathrm{Ind}_F^{\mathbb{Q}} \chi_2.$$

$$L(E \otimes \rho_1 \otimes \rho_2, s) = L(E/F, \chi, s) L(E/F, \chi\alpha/\alpha', s).$$

Hence $L(E \otimes \rho_1 \otimes \rho_2, 1) \neq 0$.
Previous theorem $\Rightarrow (E(H) \otimes \mathbb{C})^{\chi} = 0$. $\square$

Remark on Heegner points

When the real quadratic field F is replaced by an imaginary quadratic field K, the above corollary can be proved much more directly, using *Heegner points*.

Theorem (Gross-Zagier, Kolyvagin, Zhang, Bertolini-D, Longo, Nekovar, ...)

Let $L(E/K, \chi, s)$ denote the Hasse-Weil L-series of E/K, twisted by χ. Then

1. *If $L(E/K, \chi, 1) \neq 0$, then $(E(H) \otimes \mathbb{C})^{\chi} = 0$.*
2. *If $\mathrm{ord}_{s=1} L(E/K, \chi, s) = 1$, then $\dim_{\mathbb{C}}(E(H) \otimes \mathbb{C})^{\chi} = 1$.*

Stark-Heegner points attached to real quadratic fields

Motivating question: Are there structures analogous to Heegner points, when K is replaced by a *real quadratic* field?

It was this question that motivated the article

Integration on $\mathcal{H}_p \times \mathcal{H}$ and arithmetic applications, Ann. of Math. (2) 154 (2001)

in which a collection of *Stark-Heegner points*, conjecturally defined over ring class fields of real quadratic fields, were constructed.

A conditional result

Theorem (Bertolini-Dasgupta-D and Longo-Rotger-Vigni)

Assume the conjectures on Stark-Heegner points attached to the real quadratic field F (in a stronger, more precise form given in Samit Dasgupta's PhD thesis). Then

$$L(E/F,\chi,1) \neq 0 \implies (E(H) \otimes \mathbb{C})^{\chi} = 0,$$

for all ring class $\chi : \mathrm{Gal}(H/F) \longrightarrow \mathbb{C}^{\times}$.

The main interest of this result lies in the explicit connection that it draws between

1. explicit class field theory for real quadratic fields;
2. certain concrete cases of the BSD conjecture.

Euler systems and Stark-Heegner points

$$F = \text{real quadratic field}, \quad \chi : \mathrm{Gal}(H/F) \longrightarrow \mathbb{C}^\times.$$

Stark-Heegner point:

$$P_\chi \overset{?}{\in} (E(H) \otimes \mathbb{C})^\chi.$$

Question: What relation is there between the Stark-Heegner point P_χ and the class $\kappa(2,1,1)$ attached to $\rho := \mathrm{Ind}_F^{\mathbb{Q}} \chi$?

A caveat

A lot still needs to be done!

Thank you for your attention.