Genus 3 covers of elliptic curves

Davide Lombardo, Elisa Lorenzo-García, Jeroen Sijsling

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1. \( \text{Jac}(C) \sim E \times E_2 \times E_3 \), or
2. \( \text{Jac}(C) \sim E \times \text{Jac}(X) \) with \( X \) of genus 2
Goal

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2. Find $E_2$, $E_3$ or $X$. 

Remark
Finding $E_2$, $E_3$ is as hard as finding $E$, and we know how to do that.
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3. Prym-like varieties
What is known: split degree 2

Theorem (Ritzenthaler–Romagny)

Suppose

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Galois cases

- Suppose \( C \rightarrow E \) is Galois with "large" automorphism group – i.e. \( D_4, Q_8, S_3 \). Then \( \text{Jac}(C) \) is the product of three elliptic curves.
- This is not necessarily the case if the automorphism group of the covering is \( \mathbb{Z}/2\mathbb{Z} \).
- When the group is \( \mathbb{Z}/3\mathbb{Z} \), the abelian surface has QM.
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Question

Let $\Theta$ be the theta divisor of $\text{Jac}(C)$. What is the degree of the polarization $\nu_A^* A \Theta$?
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Question

Is the isogeny defined over the same field as \( C \to E \)?
From $C \to E$ determine a period matrix of $C$, hence of $A$. 

Genus 3 covers
Reconstruction: analytic approach

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- Determine an isogeny $A \rightarrow A'$ with $A'$ principally polarized
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- Write down the period matrix of $A' = \text{Jac}(X)$
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- Determine an isogeny $A \to A'$ with $A'$ principally polarized
- Write down the period matrix of $A' = \text{Jac}(X)$
- Reconstruct $X$ from $A'$ (Guàrdia)