HILBERT'S TENTH PROBLEM

(1) By a *computable function*, we mean a function $F : \mathbb{Z} \to \mathbb{N}$ that can be simulated by an algorithm. Show that there exists a polynomial $p(t, \vec{x})$ such that for any computable function $F : \mathbb{Z} \to \mathbb{N}$, there exists $a \in \mathbb{Z}$ such that $p(a, \vec{x}) = 0$ has a solution $\vec{x} \in \mathbb{Z}^n$, but no solution with $\max |x_i| < F(a)$.

(2) Recall *Mazur's conjecture*, which states that if X is a variety over $\mathbb{Q}$, then the topological closure of $X(\mathbb{Q})$ in $X(\mathbb{R})$ has at most finitely many connected components. Show that if $\mathbb{Z}$ is diophantine in $\mathbb{Q}$, then Mazur's conjecture is false.

(3) Recall that a set $A \subseteq R^n$ is *diophantine* if there exists a polynomial $p(\vec{t}, \vec{x}) \in R[t_1, \ldots, t_n, x_1, \ldots, x_m]$ such that

$$A = \{\vec{a} \in R^n : (\exists \vec{x} \in R^m)(p(\vec{a}, \vec{x}) = 0)\}.$$

Having a large dictionary of diophantine sets was essential in proving the DPRM theorem, and it will be useful in determining, for example, whether $\mathbb{Z}$ is diophantine in $\mathbb{Q}$. Show that the following sets are diophantine:

(a) The sum of two diophantine sets $A, B \subseteq \mathbb{Z}^n$ (where $A + B := \{a + b : a \in A, b \in B\}$).
(b) The intersection (or union) of two diophantine sets $A, B \subseteq \mathbb{Z}^n$.
(c) The set of nonzero integers in $\mathbb{Z}$.
(d) The set $\{(x, y) : x \leq y\}$ in $\mathbb{Z}^2$.
(e) The set $\{(x, y, z) : \gcd(x, y) = z\}$ in $\mathbb{Z}^3$.
(f) [1]The set of nonsquare integers in $\mathbb{Z}$.
(g) [2]The localization ring $\mathbb{Z}_{(p)}$ in $\mathbb{Q}$ for a fixed prime p.

[1]These last two problems are actually pretty hard. See Theorem 1.1 of *The set of nonsquares in a number field is diophantine* (Poonen).

[2]See Lemma 2.5 of *Characterizing integers among rational numbers with a universal-existential formula* (Poonen), if you don't care about the size of the defining equation, or Proposition 8 of *Defining $\mathbb{Z}$ in $\mathbb{Q}$* (Koenigsmann) for a simpler formula.