GEOMETRY OF NUMBERS

We make the Delone-Fadeev correspondence explicit. We will construct a cubic ring R from the $\mathbb{Z}$-basis $1, W, T$.

(1) Show that one can pick a new $\mathbb{Z}$-basis $1, \omega, \theta$, satisfying

$$\omega\theta = n, \quad n \in \mathbb{Z}.$$

We call $\{1, \omega, \theta\}$ the *normalized basis*. This process of choosing ω and θ is equivalent to picking a $\mathbb{Z}$-basis of $R/\mathbb{Z} \cong \mathbb{Z}^2$.

(2) This normalized basis further satisfies

$$\omega^2 = m - b\omega + a\theta, \quad m, b, a \in \mathbb{Z}$$
$$\theta^2 = \ell - d\omega + c\theta, \quad \ell, d, c \in \mathbb{Z}.$$

What are the additional conditions that need to be imposed for R to be a cubic ring? (These conditions should be imposed on n, m, ℓ to see the Delone-Fadeev correspondence.)

(3) Now, we have the set bijection

$$\{\text{cubic rings with a choice of basis of } R/\mathbb{Z}\} \xleftrightarrow{1:1} \{(a, b, c, d) \in \mathbb{Z}^4\}$$

Show that this is a discriminant-preserving bijection.

(4) Finally, convince yourself that the statement of the Delone-Fadeev correspondence given in the lecture is correct.