GALOIS REPRESENTATIONS & L-FUNCTIONS

- register
- admit people
- boards & onenote

Framework ($\supseteq$ Fermat, Langlands, BSD, ...)

Varieties $\rightsquigarrow$ Galois representations $\rightsquigarrow$ L-functions $\leftarrow$ This course

Modular & automorphic forms $\leftarrow$ Siksek's course

Plan:

$\bullet$ $\rightsquigarrow$	trivial rep. of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$	$\rightsquigarrow$ Riemann $\zeta(s)$		need number fields, Frobenius, inertia
roots of $f(x) \in \mathbb{Q}[x]$ $\rightsquigarrow$	perm. reps of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$	$\rightsquigarrow$ Dedekind $\zeta_K(s)$		
	irr. pieces	$\rightsquigarrow$ Artin $L(s)$		representation theory
ell. curves $\rightsquigarrow$	$V_\ell E$	$\rightsquigarrow$ $L(E,s)$		elliptic curves, local fields.

§ Riemann $\zeta(s)$

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \left(1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{8^s}+\ldots\right)\left(1+\frac{1}{3^s}+\ldots\right)\cdots$$

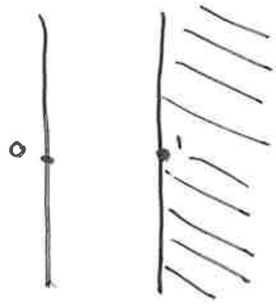
$$= \prod_p \frac{1}{1-p^{-s}}$$

Encodes distribution of primes, e.g.

$$\sum_{n\geq 1}\frac{1}{n} = \infty \implies \exists \infty \text{ primes.}$$

Viewed a fnc. of $\mathbb{C}$-variable $s = \sigma + it$

$$\left|\frac{1}{n^s}\right| = \frac{1}{n^\sigma} \implies \text{converges for } \operatorname{Re} s > 1$$



Thm (Riemann) $\zeta(s)$ has meromorphic continuation to $\mathbb{C}$, pole at $s=1$ simple, residue $=1$ and no other poles.

The completed ζ-function

poles at $s=0$ and $s=1$

$$\zeta^*(s) = \frac{1}{\pi^{s/2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

satisfies functional equation

$$\zeta^*(1-s) = \zeta^*(s)$$

Proof Poisson summation formula:

$f(n) : \mathbb{R} \longrightarrow \mathbb{C}$ $\quad$ (C^2, $|f+f''| = O\left(\frac{1}{|n|^{r+1}}\right)$ some $r>1$)

$$\hat{f}(m) = \int_0^\infty e^{2\pi i n m} f(n)\, dn \qquad \text{Fourier transform}$$

Then $\displaystyle\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)$

Apply to $f(n) = e^{-\pi x n^2}$

$$\Theta(x) := \sum_{n\in\mathbb{Z}} \overbrace{e^{-\pi x n^2}}^{f(n)} \overset{\text{Poisson}}{=} \sum_{m\in\mathbb{Z}} \frac{1}{\sqrt{x}} e^{-\frac{\pi}{x} m^2} = \frac{1}{\sqrt{x}} \Theta(x) \qquad (*)$$

Θ - function (Jacobi)

Back to ζ^*:

$$\Gamma(s) = \int_0^\infty x^s e^{-x} \frac{dx}{x} \qquad \text{Mellin transform of } e^{-x} \qquad [\Gamma(s+1) = s\Gamma(s)]$$

$$\zeta^*(2s) = \frac{1}{\pi^s} \Gamma(s) \underbrace{\zeta(2s)}_{\sum \frac{1}{n^{2s}}} = \int_0^\infty \sum_{n=1}^\infty \frac{x^s}{\pi^s n^{2s}} e^{-x}\, dx \overset{x \mapsto \pi n^2 x}{=}$$

$$= \text{Mellin tranform of } \sum_{n=1}^\infty e^{-\pi n^2 x} = \frac{\Theta(x) - 1}{2}$$

Break $\int_0^\infty = \int_0^1 + \int_1^\infty$, replace $x \mapsto \frac{1}{x}$ in 1st one using $(*)$ $\Rightarrow$

$$\zeta^*(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \frac{\Theta(x)-1}{2}\left(x^{s/2} + x^{\frac{1-s}{2}}\right)\frac{dx}{x}$$ ← converges everywhere, symmetric $s \leftrightarrow 1-s$

■

§ Conjectures

Def An L-function is a Dirichlet series

$$L(s) = \sum_{n=1}^\infty \frac{a_n}{n^s} \qquad a_n \in \mathbb{C}, \quad a_n = O(n^r) \text{ some } r$$

($\Rightarrow$ converges on $\mathrm{Re}\, s > r+1$)

It has an Euler product and degree d if

$$L(s) = \prod_p \frac{1}{F_p(p^{-s})} \qquad F_p(t) \in \mathbb{C}[t], \text{ degree} \le d,$$

$= d$ for almost all p.

All our L-functions will be of this form and are conjectured to

(A) Have mer. cont. to $\mathbb{C}$ with fin. many poles (usually none)

(B) Fun. eq.: $\exists$ weight k, sign w, conductor N,

Γ-factor $\gamma(s) = \Gamma\left(\frac{s+\lambda_1}{2}\right) \cdots \Gamma\left(\frac{s+\lambda_d}{2}\right)$ s.t.

$$L^*(s) = \left(\frac{N}{\pi^d}\right)^{s/2} \gamma(s)\, L(s)$$

satisfies

$$L^*(s) = w \cdot \bar{L}^*(k-s) \qquad \left[\bar{L}(s) = \sum \bar{a}_n n^{-s}\right]$$

(C) Riemann Hypothesis All non-trivial zeroes lie on $\mathrm{Re}\, s = \frac{k}{2}$

← not known, ever

(D) Special value conjectures on $L(n)$ for $n \in \mathbb{Z}$.

Rmks

- If $L(s)$ satisfies $(A)+(B)$, say with no poles, as before

$$L^*(s) = \int_1^\infty \Theta(\sqrt{N}\cdot x)\,(x^{\frac{s}{2}} + x^{\frac{k-s}{2}})\,\frac{dx}{x} \;;$$

$$\Theta(x) = \sum_{n=1}^{\infty} a_n\, \phi_\gamma(n,x)$$

depends only on $\gamma(s)$, decays exp. with n, e.g. $e^{-\pi n^2 x}$ for $\gamma = \Gamma(\frac{s}{2})$

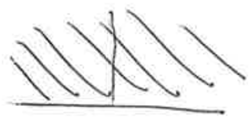
In fact, $B \Leftrightarrow$

$$\Theta\left(\frac{1}{Nx}\right) = w\cdot\Theta(x) \qquad (**)$$

Gives a way to compute L-fncs numerically (needs $\sim\sqrt{N}$ terms)

- There are "modular forms" [technically, newforms of wt k, level N, w-eigenforms for the Atkin-Lehner involution]

$$f : \{z\in\mathbb{C} \mid \operatorname{Im} z > 0\} \longrightarrow \mathbb{C}$$



such that

$$\Theta(x) = f(ix)$$

satisfies $(**)$ by definition $\Rightarrow$ their L-functions satisfy $(A)+(B)$.

- We will see two types of L-functions $L(s) = \sum \frac{a_n}{n^s}$:

(i) With an interpretation of a_n for all n

e.g. $\zeta(s)$ $\quad a_n = 1$

$L(\chi, s)$ $\quad a_n = \chi(n)$ (Dirichlet)

$\zeta_K(s)$ $\quad a_n = \#\text{ideals of norm } n \text{ in } \mathcal{O}_K$ (Dedekind)

$\Rightarrow$ Generally know how to prove $(A)+(B)$

(ii) Only defined by an Euler product

e.g. $L(\rho, s)$ Artin

$L(E, s)$ ell. curves

$\Rightarrow$ Never can prove (A) + (B) directly, only by reducing to (i).

§ Dedekind ζ-functions

K number field, $[K:\mathbb{Q}] = d$

$K \cong \mathbb{Q}^d$ as v. space

$\mathcal{O} = \mathcal{O}_K$ ring of integers

$\mathcal{O} \cong \mathbb{Z}^d$ as ab. group

$I \subseteq \mathcal{O}$ ideal $\neq 0$ $\rightsquigarrow$ $NI = (\mathcal{O}_K : I)$ norm, $< \infty$.

$$N(IJ) = NI \cdot NJ$$

$$N(n\mathcal{O}) = n^d \quad \text{for } n \in \mathbb{N}.$$

I unique product of prime ideals

$$I = \prod_{i=1}^{r} \mathfrak{p}_i^{n_i}$$

$\mathcal{O}/\mathfrak{p}_i$ finite domain $\Rightarrow$ field $\mathbb{F}_{p^r}$

In particular, for $I = (p)$ p prime $\in \mathbb{N}$

$$(p) = \prod_{i=1}^{r} \mathfrak{p}_i^{e_i}$$

$\mathfrak{p}_i$ = primes above p

e_i = ramification indices

$f_i = [\mathcal{O}/\mathfrak{p}_i : \mathbb{F}_p]$ residue degrees

Take norms $\Rightarrow$ $p^d = \prod (p^{f_i})^{e_i}$

$$\Rightarrow \quad d = \sum_{i=1}^{r} e_i f_i$$

Rmk

- $p \nmid \Delta_K \iff$ all $e_i = 1$ (p is unramified)
- If $K/\mathbb{Q}$ Galois, $e_1 = \dots = e_r$, $f_1 = \dots = f_r$ (Gal(K/Q permutes $\mathfrak{p}_i$ transitively)

In practice;

Thm (Kummer-Dedekind) $K = \mathbb{Q}[x]/(g(x))$ $\quad g \in \mathbb{Z}[x]$ monic.

Then
- $\Delta_K \mid \Delta_g$
- for all $p \nmid \Delta_g$, say $(p) = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ we have

$$g(x) \equiv g_1 \cdots g_r \mod p, \quad \deg g_i = f_i.$$

Def The Dedekind ζ-function of K

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \qquad a_n = \#\{\text{ideals of norm } n \text{ in } \mathcal{O}_K\}$$

$$= \sum_{\substack{I \subseteq \mathcal{O}_K \\ I \neq 0}} \frac{1}{(NI)^s} = \prod_{\substack{\mathfrak{p} \\ \text{prime} \\ \text{ideal}}} \frac{1}{1-(N\mathfrak{p})^{-s}}$$

$$\overset{\text{Exc}}{=} \prod_{\substack{p \\ \text{prime of } \mathbb{Z}}} \frac{1}{F_p(p^{-s})}$$

$F_p \in \mathbb{Z}[x]$

degree d for $p \nmid \Delta_K$

$< d$ for $p \mid \Delta_K$

degree d L-function.

Ex $\zeta_{\mathbb{Q}}(s) =$ Riemann $\zeta(s)$

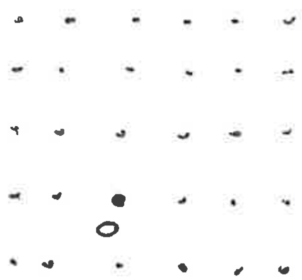
Ex $K = \mathbb{Q}(i) = \mathbb{Q}[x]/(x^2+1)$

$\mathcal{O} = \mathbb{Z}[i]$ Gaussian integers,

Euclidean $\Rightarrow$ PID $\Rightarrow$

every ideal $I = (m+ni)$

$NI = m^2+n^2$



$\mathbb{Z}[i]$

$\mathcal{O}^\times = \{\pm 1, \pm i\}$ units

prime $p = 2$ $\quad (2) = (1+i)^2$ $\quad$ ramifies

Kummer-Dedekind for $g(x) = x^2+1$ $\quad (\Delta_g = -4)$

$\Rightarrow$ all $p \neq 2$ unramified.

$p \equiv 3 \mod 4 \Rightarrow x^2+1$ irr. mod p. $\Rightarrow (p) = \mathfrak{P}_1$ $f_1 = 2$ (inert)

$p \equiv 1 \mod 4 \Rightarrow -1 \in (\mathbb{F}_p^\times)^2 \Rightarrow (p) = \mathfrak{P}_1 \mathfrak{P}_2$ $f_1 = f_2 = 1$ (split)

$\mathfrak{P}_1 = (a+bi)$

$\Leftrightarrow p = a^2+b^2$

e.g. $5 = 1^2+2^2$, $13 = 2^2+3^2$, ...

primes of $\mathbb{Z}[i]$	$(1+i)$	(3)	$(2+i)\,(2-i)$	(7)	(11)	$(2+3i)\,(2-3i)$	
primes of $\mathbb{Z}$	2	3	5	7	11	13	...
	ramifies	inert	split.				

Now

$$\zeta_{\mathbb{Q}(i)}(s) = \sum_{\substack{I \subseteq \mathbb{Z}[i] \\ I \neq 0}} \frac{1}{NI^s} = \frac{1}{4} \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0,0\}} \frac{1}{(m^2+n^2)^s}$$

$I = (m+ni)$
generator unique up to units $\pm 1, \pm i$

As for Riemann ζ,

$$\frac{2^s}{\pi^s} \Gamma(s) \zeta_K(s) = \text{Mellin transform of } \frac{\Theta_K(s) - 1}{4},$$

$$\Theta_K(s) = \sum_{m,n \in \mathbb{Z}} e^{-\pi(m^2+n^2)x} = \sum_{m \in \mathbb{Z}} e^{-\pi m^2 x} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x}$$

$$= \frac{1}{\sqrt{x}} \frac{1}{\sqrt{x}} \Theta_K\left(\frac{1}{s}\right)$$

$\Rightarrow$ mer. cont, fun. eq. for $\zeta_K(s)$.

In general: Poisson summation

$V = \mathbb{R}^d$, $f : V \longrightarrow \mathbb{C}$ (well-decaying)

V^* dual v.space,

$\hat{f} : V^* \longrightarrow \mathbb{C}$ $\qquad \hat{f}(\underline{m}) = \int_V e^{-2\pi i \langle \underline{m}, \underline{n} \rangle} f(\underline{n})\, d\underline{n}$

$\Gamma \subseteq V$ rk d lattice, Γ^* dual.

$$\sum_{\underline{n} \in \Gamma} f(\underline{n}) = \frac{1}{\mathrm{vol}(V/\Gamma)} \sum_{\underline{m} \in \Gamma^*} \hat{f}(\underline{m})$$

Compare

$$\sum_{I \neq 0} \frac{1}{(N I)^s} \quad \text{to} \quad \sum_{\substack{\alpha \in \mathcal{O} \\ \alpha \neq 0}} \frac{1}{(N\alpha)^s}$$

← involves $h = \# \frac{\text{ideals}}{\text{principal ideals}}$

← and units & roots of unity

Poisson summation $\Rightarrow$

Thm K number field of degree $d = r_1 + 2r_2$ (r_1 = # $K \hookrightarrow \mathbb{R}$, r_2 = # complex embeddings)

$\zeta_K(s)$ meromorphic, simple pole at $s=1$ with residue

$$= \frac{2^{r_1} \cdot (2\pi)^{r_2} \cdot h \cdot R}{\#\text{roots of unity in } K \cdot \sqrt{|\Delta_K|}}, \quad \text{and no other poles;}$$

$$\zeta_K^*(s) = \Big(\underbrace{\frac{|\Delta_K|}{\pi^d}}_{N}\Big)^{s/2} \overbrace{\Gamma\left(\tfrac{s}{2}\right)^{r_1+r_2} \Gamma\left(\tfrac{s+1}{2}\right)^{r_2}}^{\gamma(s)} \zeta_K(s) \text{ satisfies fun. eq.}$$

$$\zeta_K^*(1-s) = \zeta_K^*(s).$$

weight $k=1$ $\qquad$ sign $w=1$

Exc $K/\mathbb{Q}$ Galois, degree d. Then $\exists \infty$ primes that split completely in K ($e=f=1$, $r=d$); in fact, they have density $\frac{1}{d}$. [MO 218759]

__Ex__ $K = \mathbb{Q}(i)$

$$\zeta_{\mathbb{Q}(i)}(s) = \prod_{\mathfrak{p} \subseteq \mathbb{Z}[i]} \frac{1}{1 - N\mathfrak{p}^{-s}} = \prod_{p} \frac{1}{\prod_{\mathfrak{p} | p} (1 - N\mathfrak{p}^{-s})}$$

$$= \frac{1}{1-2^{-s}} \; \frac{1}{1-9^{-s}} \; \frac{1}{(1-5^{-s})(1-5^{-s})} \cdots$$

2 ramifed $\qquad$ 3 inert

$$= \prod_{p} \frac{1}{F_p(p^{-s})} \qquad F_p(T) = \begin{cases} (1-T)\cdot 1 & p = 2 \\ (1-T)(1-T) & p \equiv 1 \bmod 4 \\ \underbrace{(1-T)(1+T)}_{= 1-T^2} & p \equiv 3 \bmod 4 \end{cases}$$

__Question__ Is $\zeta_{\mathbb{Q}(i)}(s) = \zeta(s) \times$ interesting L-function $\sum \frac{a_n}{n^s}$ with finitely many poles ?

__Answer__ Yes : $a_n = \left(\frac{n}{4}\right)$, $L(s) = \sum \frac{a_n}{n^s}$ Dirichlet L-function, no poles.

Next steps :

1) $\zeta_{\mathbb{Q}(\zeta_n)}(s) = \prod$ Dirichlet L-functions "ad hoc"

2) K any number field

$\zeta_K(s) = \prod L(\rho, s)$ Artin L-functions rep. theory