§ Artin representations

Def G finite group. A d-dimensional (complex) representation of G is a group hom.

$$\rho : G \longrightarrow \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ GL_d(\mathbb{C})}}{GL(V)} \qquad V \quad \mathbb{C}\text{-v. space of dim } d$$

When K/F fin. Gal. ext. of number fields,

$$\rho : \quad Gal(K/F) \longrightarrow GL(V)$$
$$Gal(\bar{F}/F) \nearrow$$

is an Artin representation (over F). ← sometimes V is called the rep. of G, if $G \circlearrowright V$ is understood

Ex 1-dim Artin rep. $=$ gp. hom $Gal(K/F) \longrightarrow \mathbb{C}^\times$ "1-dim character"

$[=$ Cont. hom. $\underset{\text{profinite top.}}{Gal(\bar{F}/F)} \longrightarrow \underset{\text{discrete top.}}{\mathbb{C}^\times}]$

Ex $F = \mathbb{Q}$

$K = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$

$$\Delta : \quad Gal(K/F) \cong S_3 \longrightarrow GL_2(\mathbb{C})$$
$$(123) \longmapsto \text{rot. by } 2\pi/3 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$
$$(23) \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we are "representing" $Gal(K/F)$ as a group of matrices

is a 2-dim. Artin rep. over $\mathbb{Q}$.

Def $\rho : Gal(K/F) \to GL(V)$ Artin representation. The Artin L-function

$$L(\rho, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} := \prod_{\substack{\mathfrak{p} \text{ prime} \\ \text{of } F}} \frac{1}{F_\mathfrak{p}(N\mathfrak{p}^{-s})} \quad ; \quad F_\mathfrak{p}(T) = \det\left(1 - \rho(Frob_\mathfrak{p}^{-1})T \,\middle|\, V^{I_\mathfrak{p}}\right)$$

⚠ Have no interpretation for a_n for composite n

for unramified primes has degree d, depends only on conj. class of $Frob_\mathfrak{p} \in Gal(K/F)$

inertia invariants $\{v \in V \mid \sigma(v) = v \;\; \forall \sigma \in I_\mathfrak{p}\}$
$= V$ if $\mathfrak{p}$ unramified in K/F

Exc (Advised) • This is independent of the choice of $I_\mathfrak{p}$, $Frob_\mathfrak{p}$.

• Writing $L(\rho, s) = \prod_{\substack{p \text{ prime} \\ \text{of } \mathbb{Q}}} \left[\prod_{\mathfrak{p} | p} \frac{1}{F_\mathfrak{p}(N\mathfrak{p}^{-s})} \right]$

check that $L(\rho, s)$ is an L-fnc of degree $\dim\rho \cdot [F:\mathbb{Q}]$.

Ex $\quad \Delta: \mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}) \cong S_3 \longrightarrow GL(V)$, $\quad \dim V = 2$.

$$\begin{array}{lcll}
\mathrm{id} & \longmapsto & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{char pol}_s. \ (1-T)^2 \\
\text{2-cycles} & \longmapsto & \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{char.poly } (1-T)(1+T) \\
\text{3-cycles} & \longmapsto & \sim \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix} & \text{charpoly } 1+T+T^2
\end{array}$$

$$L(\Delta, s) = \underset{\substack{I_2 = C_3 \\ V^{I_2}=0}}{1} \cdot \underset{\substack{I_3 = S_3 \\ V^{I_3}=0}}{1} \cdot \underset{\mathrm{Frob}_5 \text{ 2-cycle}}{\frac{1}{(1-5^{-s})(1+5^{-s})}} \cdot \underset{\mathrm{Frob}_7 \text{ 3-cycle}}{\frac{1}{1+7^{-s}+7^{-2s}}} \cdot \ldots$$

Exc $\quad$ Find Δ and $L(\Delta, s)$ on LMFDB.

§ 1-dimensional Artin representations

Thm $\quad$ There is a bijection $\quad \leftarrow$ actually iso of groups • on the left, ⊗ on the right

$$\{\text{Dirichlet characters}\} \longleftrightarrow \{\text{1-dim Artin reps } \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^\times\}$$
$$\chi \longmapsto \rho_\chi$$

such that

(A) χ has modulus $m \iff \rho_\chi$ factors through $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ and not for smaller $n | m$

(B) $L(\chi, s) = L(\rho_\chi, s)$

Proof $\quad$ Take $\chi: (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ primitive, let

$$\rho_\chi: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \overset{\text{can.}}{\cong} (\mathbb{Z}/m\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$$
$$\sigma: \zeta_m \mapsto \zeta_m^a \longmapsto a^{-1} \longmapsto \chi(a)^{-1}$$

(A) is clear.

Conversely, if $\rho: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^\times$ 1-dim. Artin rep.

$N := \ker \rho \quad$ Normal

$K := \overline{\mathbb{Q}}^N \quad$ Galois

$\mathrm{Gal}(K/\mathbb{Q}) \cong \mathrm{Im}\rho < \mathbb{C}^\times$ abelian, so $K \subseteq \mathbb{Q}(\zeta_m)$ by Kronecker-Weber, some m

Therefore $\rho \cong \rho_\chi$ for some χ.

(B) Compare L-functions $L(\chi,s)$ and $L(\rho_\chi,s)$ by local factors.

<u>$p \nmid m$</u> $\Longleftrightarrow$ p unramified in $\mathbb{Q}(\zeta_m)/\mathbb{Q}$ $\Rightarrow I_p = \{id\}$.

$$F_p(T)_{LHS} = 1 - \chi(p)T$$

$$F_p(T)_{RHS} = 1 - \rho_\chi(\mathrm{Frob}_p^{-1})T \qquad \text{under } \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$$

$$\begin{array}{ccc} (\zeta_m \mapsto \zeta_m^{p^{-1}}) & \longmapsfrom & p \\ \| & & \\ \mathrm{Frob}_p^{-1} & & \end{array}$$

$$= 1 - \chi(p)T$$

<u>$p \mid m$</u> $\quad F_p(T)_{LHS} = 1$

$$m = m_0 p^k \qquad \begin{array}{c} \mathbb{Q}(\zeta_m) \\ I_p \,| \\ \mathbb{Q}(\zeta_{m_0}) \\ | \\ \mathbb{Q} \end{array} \Bigg) (\mathbb{Z}/m\mathbb{Z})^\times$$

χ primitive $\Rightarrow$ does not factor through $(\mathbb{Z}/m_0\mathbb{Z})^\times$

$\Rightarrow \chi(I_p) \neq 1 \Rightarrow V^{I_p} = \{0\}$

$\Rightarrow F_p(T)_{RHS} = 1$ ∎

<u>Rmk</u> Same holds over any number field F:

$$\begin{array}{ccc} \text{Hecke characters } / F & \overset{1:1}{\longleftrightarrow} & \text{1-dim Artin} \\ \text{of trivial } \infty \text{ type} & & \text{reps over } F \\ \chi & \longmapsto & \rho_\chi \\ L(\chi,s) & = & L(\rho_\chi,s) \end{array}$$

<u>Proof</u> Instead of Kronecker-Weber, full force of global CFT.

§ Permutation representations & Dedekind ζ

G finite group

X G-set $\rightsquigarrow$ representation $\mathbb{C}[X]$ $\leftarrow$ basis $\{x\}_{x\in X}$ over $\mathbb{C}$, permuted by G.

Note: $\mathbb{C}[X]^G \cong \mathbb{C}^{\#\text{orbits of } G \text{ on } X}$ (*) $\leftarrow$ e.g. $x_1 \rightleftarrows x_2 \quad x_3 \to x_4 \to x_5 \to x_3$

$\mathbb{C}[X]^G = \langle x_1+x_2,\ x_3+x_4+x_5\rangle$

Thm M/F finite, K Galois closure

$X := \{F \text{ embeddings } M \hookrightarrow K\}$

$V = \mathbb{C}[X_{M/F}]$ associated perm. rep. of $G = \mathrm{Gal}(K/F)$ $\leftarrow$ $\dim V = \# X = [M:F]$

Then

(i) $L(V,s) = \zeta_M(s)$. $\leftarrow$ So Dedekind ζ-fncs are Artin L-fncs for perm. reps.

(ii) let $\mathfrak{p} \subseteq \mathcal{O}_F$ be a prime, $\mathfrak{p} = \mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_r^{e_r}$ in $\mathcal{O}_M$, res. degrees f_i. Then

$$\det\left(1-\mathrm{Frob}_{\mathfrak{p}}^{-1}T \mid V^{I_{\mathfrak{p}}}\right) = \prod_i (1-T^{f_i})$$

Proof (ii)

	K	$\tilde{\mathfrak{p}}$
G	$\mid H$	$\backslash\mid/$
	M	$\mathfrak{p}_1 \cdots \mathfrak{p}_r$
	$\mid$	$\backslash\mid/$
	F	$\mathfrak{p}$

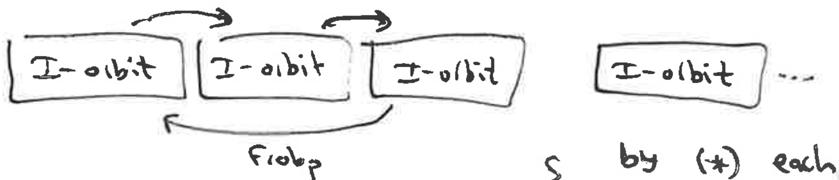
$K :=$ Galois closure of M/F

$\tilde{\mathfrak{p}} :=$ prime above $\mathfrak{p}$ in K,

$\rightsquigarrow D = D_{\tilde{\mathfrak{p}}},\ I = I_{\tilde{\mathfrak{p}}},\ \mathrm{Frob} = \mathrm{Frob}_{\tilde{\mathfrak{p}}} \in G$.

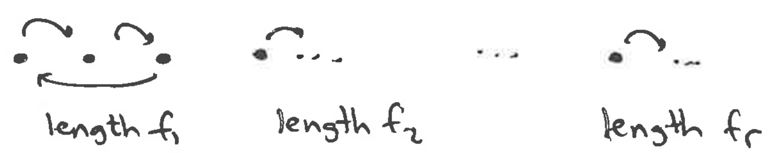
As a D-set,

$X = G/H =$ union of orbits of length $e_j f_j$, each is a union of f_j I-orbits of length e_j, cyclically permuted by Frob.



$\rightsquigarrow$ by (*) each I-orbit contributes 1 basis vector to $\mathbb{C}[X]^I$

Basis of $\mathbb{C}[X]^I$:



Frob (and Frob^{-1}) act on each orbit as

$$\underbrace{\begin{pmatrix} 0 & 0 & & 1 \\ 1 & 0 & & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & & & \vdots \end{pmatrix}}_{f_i \times f_i} \qquad \text{char. poly. } T^{f_i} - 1$$

So $\det(1 - \mathrm{Frob}_p^{-1} T \mid V^{I_p}) = \prod_i (1 - T^{f_i})$, as claimed.

(i) $$L(V,s) = \prod_{\substack{p \text{ prime} \\ \text{of } F}} \prod_{\substack{P_i | p \\ \text{of } M}} \frac{1}{1 - (N_{F|\mathbb{Q}}\, p^{-s})^{f_{P_i|p}}} = \prod_{q \text{ prime of } M} \frac{1}{1 - (N_{M|\mathbb{Q}}\, q)^{-s}}$$

$$= \zeta_M(s). \qquad \blacksquare$$

<u>Conclusion</u>

$$\begin{tikzcd}[column sep=small] K \arrow[dd, dash, "G"'] \arrow[dr, dash, "H"] & \\ & M \arrow[dl, dash] \\ F & \end{tikzcd}$$

More than we bargained for: We can get the same function $\zeta_M(s)$ from Artin reps for different fields

$$L(\mathbb{1}_H, s) = \prod_{\mathfrak{p} \subseteq \mathcal{O}_M \text{ prime}} \frac{1}{1 - N\mathfrak{p}^{-s}} = \zeta_M(s)$$

rep. of H (or of $\mathrm{Gal}(\bar{F}/M)$), i.e. Artin rep over M

$$L(\mathbb{C}[G/H], s) = \prod_{\mathfrak{p} \subseteq \mathcal{O}_F \text{ prime}} \det(\dots)^{-1} = \zeta_M(s)$$

rep. of G (or of $\mathrm{Gal}(\bar{F}/F)$), i.e. Artin rep over F

Generally, there is a functor

Induction: $\{\text{Reps of } H\} \longrightarrow \{\text{Reps of } G\}$

and it preserves L-functions : "Artin formalism"

§ Representations of finite groups

G finite group.

Representation theory $\Rightarrow$

- $\rho_1 : G \longrightarrow GL(V_1)$ reps.
  $\rho_2 : G \longrightarrow GL(V_2)$

  Def $\rho_1 \oplus \rho_2 : G \longrightarrow GL(V_1 \oplus V_2)$

  $$g \longmapsto \left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right)$$ direct sum

- ρ_1 and ρ_2 are isomorphic if $\exists\, \varphi : V_1 \xrightarrow{\cong} V_2$ s.t.

$$\begin{tikzcd} V_1 \arrow[r, "\varphi"] \arrow[d, "\rho_1(g)"'] & V_2 \arrow[d, "\rho_2(g)"] \\ V_1 \arrow[r, "\varphi"] & V_2 \end{tikzcd}$$

  commutes for all $g \in G$.

  not $\cong$ direct sum

- There are fin. many irr. reps $\rho_1, \ldots, \rho_k$ of G s.t. every rep. of G is

  $$\cong \rho_1^{\oplus n_1} \oplus \cdots \oplus \rho_k^{\oplus n_k} \quad \text{for some } n_i \geq 0; \qquad \sum_{i=1}^{k} (\dim \rho_i)^2 = |G|$$

  Ex all $\dim \rho_i = 1 \iff G$ abelian.

- Induction :

  $H < G$ index n

  $\rho : H \longrightarrow GL(V)$ $\rightsquigarrow$ $\mathrm{Ind}_H^G V = \quad \to V \mid f(hg) = \rho(h) f(g) \ \forall h \in H\}$

  d-dim rep. of H $\qquad$ $\circlearrowright$ $\qquad$ nd-dim. rep. of G

  $g \in G$ acts as
  $f(x) \mapsto f($

  $\rho = \mathbb{1} : H \longrightarrow \mathbb{C}^\times$

  $\mathrm{Ind}_H^G \mathbb{1} \cong \mathbb{C}[G/H]$.