Lecture 5

<u>Ex</u> $G = C_3 = \langle g \rangle$

$\mathbb{1} : G \longrightarrow \mathbb{C}^\times$ trivial representation

$g \longmapsto 1$

$\chi : G \longrightarrow \mathbb{C}^\times$

$g \longmapsto \zeta_3$

$\overline{\chi} : G \longrightarrow \mathbb{C}^\times$

$g \longmapsto \zeta_3^2$

non-trivial irr. reps of G

$1^2 + 1^2 + 1^2 = 3$

<u>Ex</u> $G = S_3 = \langle (123), (12) \rangle$

$\mathbb{1} : G \longrightarrow \mathbb{C}^\times$ trivial rep.

all $g \longmapsto 1$

$\mathcal{E} : G \longrightarrow \mathbb{C}^\times$ sign rep.

$(123) \longmapsto 1$

$(12) \longmapsto -1$

$\Delta : G \longrightarrow GL_2(\mathbb{C})$

$id \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$(23) \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$(123) \longmapsto \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

group of symmetries of triangle

$1^2 + 1^2 + 2^2 = \# S_3 \Rightarrow$ these are all irr. reps $\Rightarrow$

$\Rightarrow$ every rep of $S_3 \cong \mathbb{1}^{\oplus n_1} \oplus \mathcal{E}^{\oplus n_2} \oplus \Delta^{\oplus n_3}$, e.g.

$\mathbb{C}[S_3 / C_3] \cong \mathbb{1} \oplus \mathcal{E}$

$\mathbb{C}[S_3 / C_2] \cong \mathbb{1} \oplus \Delta$

$\mathbb{C}[S_3 / \{id\}] \cong \mathbb{1} \oplus \mathcal{E} \oplus \Delta \oplus \Delta$

Recall: $\mathrm{Ind}_{C_3}^{S_3} \mathbb{1} \cong \mathbb{C}[S_3 / C_3] \cong \mathbb{1} \oplus \mathcal{E}$

<u>Exc</u> $\mathrm{Ind}_{C_3}^{S_3} \chi \cong \mathrm{Ind}_{C_3}^{S_3} \overline{\chi} \cong \Delta$

§ Artin formalism

Thm $K/F/M$ number fields

K/M Galois, $G = \mathrm{Gal}(K/M)$

ρ, ρ' Artin reps of $H = \mathrm{Gal}(K/F) < G$.

$$\begin{tikzcd}
K \arrow[dd, dash, "G"'] \arrow[dr, dash, "H"] & \\
& F \arrow[dl, dash] \\
M &
\end{tikzcd}$$

Then

① $L(\rho \oplus \rho', s) = L(\rho, s)\, L(\rho', s)$ direct sums

② $L(\mathrm{Ind}_H^G \rho, s) = L(\rho, s)$ induction

Proof ① From definition

② Similar to $\rho = \mathbb{1}$, using Mackey's formula

$$\mathrm{Res}_D \mathrm{Ind}_H^G \rho = \bigoplus_{DgH \in D\backslash G/H} \mathrm{Ind}_{D \cap gHg^{-1}}^D \rho^g \qquad \blacksquare$$

Cor Artin reps $/\mathbb{Q}$ give all Artin L-functions / all number fields.

Thm (Brauer Induction) $\rho : G \longrightarrow GL(V)$ rep.

There are sgps $H_i, H_i' < G$, 1-dim reps $\chi_i : H_i \to \mathbb{C}^\times$, $\chi_i' : H_i' \to \mathbb{C}^\times$ s.t.

$$\rho \oplus \mathrm{Ind}_{H_1}^G \chi_1 \oplus \dots \cong \mathrm{Ind}_{H_1'}^G \chi_1' \oplus \dots$$ as reps of G

Ex $H = C_3$ $\mathbb{1}, \chi, \overline{\chi}$
$\wedge$
$G = S_3$ $\mathbb{1}, \varepsilon, \underline{\underline{\Delta}}$

irr. reps, say $G = \mathrm{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q})$

$\rho = \Delta$: $\Delta \cong \mathrm{Ind}_{C_3}^{S_3} \chi$ and $\underbrace{\mathrm{Ind}_{S_3}^{S_3} \mathbb{1}}_{\mathbb{1}} \oplus \Delta = \underbrace{\mathrm{Ind}_{C_2}^{S_3} \mathbb{1}}_{\mathbb{1} \oplus \Delta}$

no uniqueness in Brauer

$$L(\Delta, s) = L(\chi, s) = \frac{\zeta_{\mathbb{Q}(\sqrt[3]{2})}(s)}{\zeta(s)}$$

Cor $L(\rho, s) = \dfrac{\prod L(\chi_i', s)}{\prod L(\chi_i, s)}$ ← every Artin L-fnc is a quotient of products of Hecke L-fncs!

In particular, $L(\rho, s)$ has meromorphic cont to $\mathbb{C}$. ← possibly ∞ poles from zeroes of $L(\chi_i, s)$)

Cor If we start with Hecke L-fncs and want Artin formalism, our formula for Artin L-fncs

$$L(V,s) = \prod_{p} F_p(Np^{-s})^{-1} \quad ; \quad F_p(T) = \det(1-\mathrm{Frob}_p^{-1}T \mid V^{I_p})$$

is the only possible one.

Conj (Artin) $L(\rho,s)$ is analytic on $\mathbb{C}$ (no poles) for any $\rho \neq \mathbb{1}$ irr. Artin rep.

Ex $G = S_3$

$$\left.\begin{array}{l} \mathbb{1}, \varepsilon \rightsquigarrow \text{Hecke L-fnc} \\ \Delta = \mathrm{Ind}_{C_3}^{S_3}\chi \rightsquigarrow \text{also Hecke} \end{array}\right\} \text{Artin's Conj is true for } S_3$$

Generally:

Def G is monomial if every irr. rep. is induced from a 1-dim one.

Smallest non-monomial groups: $SL_2(\mathbb{F}_3)$, 4 gps of order 48, A_5, ...

order 24   order 60

← quite rare   ↑ all A_n, S_n, $n \geq 5$.

Thm Suppose $\rho, \sigma : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_d(\mathbb{C})$ Artin reps. Then

$$\rho \cong \sigma \iff L(\rho,s) = L(\sigma,s)$$ as anal. fncs on $\mathrm{Re}\, s > 1$,

so $L(\rho,s)$ determines ρ uniquely.

Proof $\Rightarrow$ Clear.

$\Leftarrow$ Step 1 $f(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$ any Dirichlet series, conv. for $\mathrm{Re}\, s$ large

$$a_1 = \lim_{x\to\infty} f(x)$$

$$a_2 = \lim_{x\to\infty} 2^x(f(x)-a_1)$$

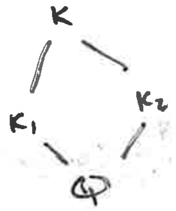
...

So a_i uniquely determined by $f(s)$ as a function.

Hence ρ, σ have the same local factors at all primes.

Step 2 Say $\rho : \mathrm{Gal}(K_1/\mathbb{Q}) \longrightarrow GL(V_1)$

$\sigma : \mathrm{Gal}(K_2/\mathbb{Q}) \longrightarrow GL(V_2)$



$\Rightarrow$ Both factor through $G = \mathrm{Gal}(K_1K_2/\mathbb{Q})$

Step 3 Rep. theory $\Rightarrow$ enough to prove for every conj. class $C \subseteq G$, $g \in C$

$\mathrm{tr}\,\rho(g) = \mathrm{tr}\,\sigma(g)$

$\nwarrow$ called character of a rep.

By Čebotarev, $\exists p$ s.t. $\mathrm{Frob}_p \in C$. (and $I_p = \{1\}$)

$F_p(\rho, T) = F_p(\sigma, T)$; same char. polys $\Rightarrow$ same traces ∎

§ Γ-factors, ε-factors and conductors

Thm $\rho : \mathrm{Gal}(K/F) \longrightarrow GL_d(\mathbb{C})$ Artin rep; $L(\rho,s)$ meromorphic on $\mathbb{C}$. Then

$$L^*(\rho,s) = \left(\frac{N_\rho}{\pi^{d_\rho}}\right)^{s/2} \gamma_\rho(s)\, L(\rho,s)$$

satisfies fun. eq.

$$L^*(\rho,s) = w_\rho\, \overline{L}(1-s).$$

for some $\overset{\textcircled{1}}{d_\rho \in \mathbb{Z}}$, $\overset{\textcircled{4}}{N_\rho \in \mathbb{Q}^\times}$, $\overset{\textcircled{2}}{\gamma_\rho(s) = \Gamma\left(\frac{s}{2}\right)^{?}\Gamma\left(\frac{s+1}{2}\right)^{?}}$, $\overset{\textcircled{3}}{w_\rho \in \mathbb{C}^\times}$, $|w_\rho| = 1$.

Proof Recall $L(\rho,s) = \dfrac{\prod L(\chi_i,s)}{\prod L(\chi_i',s)}$ some χ_i, χ_i' of dim 1

$\nwarrow$

all these satisfy fun. eqs $s \leftrightarrow 1-s$ (Hecke)

so $L(\rho,s)$ does, with $d_\rho = \sum \dim \chi_i - \sum \dim \chi_i'$, $N_\rho = \dfrac{\prod N_{\chi_i}}{\prod N_{\chi_i'}}$, etc. ∎

We do understand these invariants $\leftarrow$ We know them for Hecke characters, and extend in a unique way to be compatible with Artin formalism.

Suppose $F = \mathbb{Q}$ (else replace ρ by $\mathrm{Ind}_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \rho$), so

$$\rho : \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow GL_d(\mathbb{C}).$$

① $d_\rho = \dim \rho$ $\qquad$ = degree of the L-function.

② γ- factor

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_d(\mathbb{C})$$

complex conjugation $\longmapsto$ matrix of order 2

say d_+ eigenvalues $+1$

d_- eigenvalues -1 $\qquad$ $d_+ + d_- = d_\rho$

Then $\gamma_\rho(s) = \Gamma\left(\frac{s}{2}\right)^{d_+} \Gamma\left(\frac{s+1}{2}\right)^{d_-}$

Ex $M/\mathbb{Q}$ finite.

$\zeta_M(s) = L(\mathbb{C}[X_{M/\mathbb{Q}}],s)$ $\qquad$ $X_{M/\mathbb{Q}} = \{\text{embeddings } M \hookrightarrow \overline{\mathbb{Q}}\} \circlearrowleft \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

complex conj. fixes r_1 real embeddings,

swaps complex ones in pairs

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & & \\ & & & & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & & & & \ddots \end{pmatrix}$$ action on $\mathbb{C}[X_{M/\mathbb{Q}}]$

$\Rightarrow$ $r_1 + r_2$ eigenvalues $+1$ $\Rightarrow$ $\gamma_{\mathbb{C}[X]}(s) = \Gamma\left(\frac{s}{2}\right)^{r_1+r_2} \Gamma\left(\frac{s+1}{2}\right)^{r_2}$

r_2 eigenvalues -1

as expected for $\zeta_M(s)$.

③ Sign in the functional equation w_ρ

Very complicated!

Hecke, Tate for Hecke characters

Langlands, Deligne in general $\Bigg\}$ "Theory of ε-factors".

④ § Conductor N_ρ

$\rho : G = \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow GL(V)$, $\dim V = d$.

Def The global conductor $N = N_\rho$ is

$$N := \prod_p p^{n_p} \quad , \quad n_p = \text{local conductor exponent at } p$$

Fix p, $D = D_p < G$ decomposition gp. of some $\mathfrak{p} \mid p$ in K.

$I = I_p \triangleleft D$ inertia gp.

Then $n_p := n_{p,\mathrm{tame}} + n_{p,\mathrm{wild}}$

$n_{p,\mathrm{tame}} := d - \dim V^I$ ← missing degree for $F_p(T)$

$n_{p,\mathrm{wild}} = 0$ if $p \nmid |I|$

In general,

$$G > D \triangleright \underset{\text{inertia}}{I_0 = I} \triangleright \underset{\text{wild inertia}}{I_1 = p\text{-Sylow}(I)} \triangleright I_2 \triangleright \cdots$$

$$I_n = \{\sigma \in D \mid \sigma = \mathrm{id} \mod \mathcal{O}_K / \mathfrak{p}^{n+1}\}$$

higher ramification groups,
$= \{\mathrm{id}\}$ for n large.

$$n_{p,\mathrm{wild}} := \sum_{n \geq 1} \frac{|I_n|}{|I|} (d - \dim V^{I_n}) \in \mathbb{Z}$$

measures how badly ramified V is.

Ex ρ unramified at $p \overset{\mathrm{def}}{\iff} V^I = V \iff n_{p,\mathrm{tame}} = 0 \iff n_p = 0$.

Exc $\rho = \rho_\chi : G_\mathbb{Q} \longrightarrow \mathbb{C}^\times$ χ Dirichlet character.

Then $N_\rho = \text{modulus}(\chi)$.

Ex $M/\mathbb{Q}$ finite.

$\zeta_M(s) = L(\mathbb{C}[X_{M/\mathbb{Q}}], s)$

Then $N_{\mathbb{C}[X_{M/\mathbb{Q}}]} = |\Delta_{M/\mathbb{Q}}|$ ← Führerdiskriminantenformel gives a way to compute discriminants from Artin reps

Ex

$$\begin{tikzcd}
F = \mathbb{Q}(\overset{\zeta_3}{\overset{\Vert}{\zeta}}, \sqrt[3]{3}) \arrow[d, dash, "C_2"] \arrow[dd, dash, bend right, "S_3"'] & q \arrow[d, "\text{tot. ram.}"] & = (\pi) & \pi = \frac{1-\zeta_3}{\sqrt[3]{3}} \\
M = \mathbb{Q}(\sqrt[3]{3}) \arrow[d, dash] & 3 & & \\
\mathbb{Q} & & &
\end{tikzcd}$$

$\leftarrow v_3 = \frac{1}{2}$

$\nwarrow v_3 = \frac{1}{3}$

$\underbrace{I_1}_{\substack{3\text{-Sylow}\\ = C_3}} \triangleleft \underbrace{I = D = G}_{S_3}$ gen σ^{-1} of I_1 : $\sqrt[3]{3} \longmapsto \zeta\sqrt[3]{3}$, $1-\zeta \longmapsto 1-\zeta$

$$\sigma(\pi) = \zeta\pi$$

$$\Rightarrow v_q(\pi - \zeta\pi) = 1 + \overbrace{v_q(1-\zeta)}^{3} = 4.$$

$$\Rightarrow \sigma \equiv 1 \mod \pi^4$$
$$\not\equiv 1 \mod \pi^5$$

$$\underbrace{\cdots \{1\}}_{\{1\}} \triangleleft \underbrace{I_3 = I_2 = I_1}_{C_3} \triangleleft \underbrace{I}_{S_3}$$

Take $V = \mathbb{C}[X_{M|K}] = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$

S_3 acts naturally

S_3, C_3 have 1-dim invariants (# orbits)

$\{1\}$ has 3-dim.

$$n_{V,3} = \overbrace{3-1}^{\text{tame}} + \overbrace{\tfrac{3}{6}(3-1)}^{I_1} + \overbrace{\tfrac{3}{6}(3-1)}^{I_2} + \overbrace{\tfrac{3}{6}(3-1)}^{I_3} + 0 = 5.$$

$n_{V,p} = 0 \quad \forall p \neq 3$ as p unr. in $F|K$ $\Rightarrow I_p = \{1\}$

$\Rightarrow V$ unramified at p.

So $|\Delta_M| = N_V = 3^5$ [and $|\Delta_F| = 3^{11}$ – Exc.]

~~Finally, conductors [and ε-factors] are inductive in degree 0.~~

~~**Thm** $[K:\mathbb{Q}] = n$~~

~~$\rho_1, \rho_2 : G_K \longrightarrow GL_d(\mathbb{C})$, $\mathrm{Ind}\,\rho_1$, $\mathrm{Ind}\,\rho_2 : G_{\mathbb{Q}} \longrightarrow GL_{dn}(\mathbb{C})$. Then~~

~~$\mathrm{Norm}_{K|\mathbb{Q}} \frac{N(\rho_1)}{N(\rho_2)} = \frac{N(\mathrm{Ind}\,\rho_1)}{N(\mathrm{Ind}\,\rho_2)}$~~

~~enters L-fac $L(\rho_1, s)$~~

~~**Cor** [take $\rho_2 = 1\!\!1 \oplus \cdots \oplus 1\!\!1$ d times] $N(\mathrm{Ind}\,\rho_1) = \mathrm{Norm}_{K|\mathbb{Q}} N(\rho_1) \cdot |\Delta_K|^d$~~