## § Zeta functions / $\mathbb{F}_p$

$X / \mathbb{F}_p$ variety

$x \in X$ closed point $\Leftrightarrow$ res. field $k_x$ finite $\Leftrightarrow$ $x \in X(\mathbb{F}_{p^n})$ some $n$.

**Def** Zeta function

$$Z_X(T) = \exp\Big( \sum_{n \geq 1} \frac{\# X(\mathbb{F}_{p^n})}{n} T^n \Big) \overset{\text{Exc}}{=} \prod_{\substack{x \text{ closed} \\ \text{point}}} \frac{1}{1 - T^{(k_x : \mathbb{F}_p)}} \in \mathbb{Q}[[T]]$$

**Rmk** $X = \coprod_i X_i \Rightarrow Z_X(T) = \prod_{i \in I} Z_{X_i}(T)$ $\quad$ ($|I| = \infty$ allowed)

**Ex** $X = \{pt\}$

$$Z_X(T) = \exp\Big( \sum_{n \geq 1} \frac{1}{n} T^n \Big) = \exp(-\log(1-T)) = \frac{1}{1-T}$$

**Ex** $X = \mathbb{A}^1 / \mathbb{F}_p$


$X(\mathbb{F}_{p^n}) = \mathbb{F}_{p^n}$

$$Z_X(T) = \exp\Big( \sum_{n \geq 1} \frac{p^n}{n} T^n \Big) = \exp(\log(1-pT)) = \frac{1}{1-pT}$$

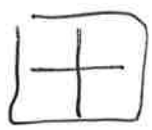
Similarly

$$Z_{\mathbb{A}^m}(T) = \frac{1}{1 - p^m T}$$

**Ex**

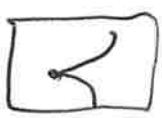
$X = \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$

$Z_X(T) = \frac{1}{(1-T)(1-pT)}$



$X : xy = 0 \subseteq \mathbb{A}^2 \quad = \mathbb{A}^1 \amalg \mathbb{A}^1 \setminus \{pt\}$

$Z_X(T) = \frac{1-T}{(1-pT)^2}$

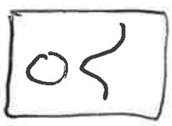


$X : y^2 = x^3 \subseteq \mathbb{A}^2$

$\mathbb{A}^1 \xrightarrow{1:1} X$

$t \longmapsto (t^2 : t^3)$

$Z_X(T) = \frac{1}{1-pT}$ $\quad \leftarrow$ $Z_X$ does not care too much about geometry



$X : y^2 = x^3 + ax + b \subseteq \mathbb{P}^2_{\mathbb{F}_p}$ ell. curve

$$Z_X(T) = \frac{1 - tT + pT^2}{(1-T)(1-pT)} \qquad |t| < 2\sqrt{p} \qquad \text{Hasse-Weil.}$$

**Rmk** $x^2 y = 0 \subseteq \mathbb{A}^2$ and $xy = 0$ in $\mathbb{A}^2$ have the same Z-function; generally $Z_X = Z_{X^{red}}$.

**Thm** (Dwork) $Z_X(T)$ is a rational fnc. of $T$ for any $X / \mathbb{F}_p$ of finite type.

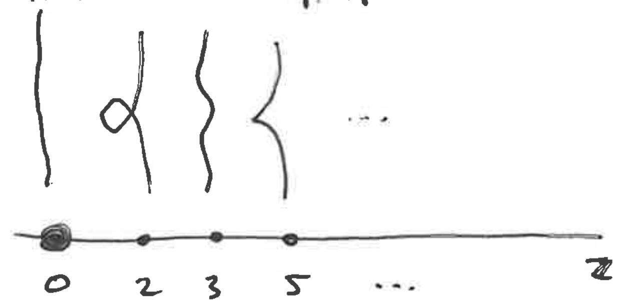
## § Zeta functions / $\mathbb{Z}$

$\mathfrak{X}/\mathbb{Z}$ of finite type, say

$$\begin{cases} f_1(x_1,\dots,x_n)=0 \\ \dots \\ f_m(x_1,\dots,x_n)=0 \end{cases} \quad \text{in } \mathbb{A}^n_{\mathbb{Z}} \text{ or } \mathbb{P}^{n-1}_{\mathbb{Z}}$$

e.g. $\mathbb{A}^n_{\mathbb{Z}}$, $\mathbb{P}^n_{\mathbb{Z}}$, $x^2+1=0\ /\mathbb{Z}$, $y^2=x^3+1\ /\mathbb{Z}$, or any $X/\mathbb{F}_p$ (take $p=0$ as one of eqns)

generic fibre $\mathfrak{X}_{\mathbb{Q}}/\mathbb{Q}$ special fibres $\mathfrak{X}_p/\mathbb{F}_p$ ← take defining eqns $/\mathbb{Q}$ or $/\mathbb{F}_p$.




As before,

$x\in\mathfrak{X}$ closed point $\iff$ residue field $k_x$ is finite $\iff$ $x\in\mathfrak{X}(\mathbb{F}_{p^n})$ for some $n$.

**Def** Zeta function

$$\zeta_{\mathfrak{X}}(s) = \prod_{\substack{x \text{ closed} \\ \text{point of } \mathfrak{X}}} \frac{1}{1-|k_x|^{-s}}$$

← again multiplicative for disjoint unions

$$= \prod_p Z_{\mathfrak{X}_p}(p^{-s}) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

converges for $\operatorname{Re} s > \dim\mathfrak{X}$

**BIG CONJECTURE** $\zeta_{\mathfrak{X}}(s)$ has meromorphic continuation to $\mathbb{C}$

↖ very hard, e.g. for $y^2=x^3+ax+b\ /\mathbb{Z}$ this is Shimura–Taniyama conj. (Wiles et al.)

Ex $X/\mathbb{F}_p$ any

$\zeta_X(s) = Z_X(p^{-s})$ meromorphic by Dwork.

So assume $\mathcal{X}_{\mathbb{Q}} \neq \phi$ (i.e. $\mathcal{X}$ "lives in characteristic 0")

Ex $\mathcal{X} = \operatorname{Spec} \mathbb{Z}$

closed pts = non-zero prime ideals $(p) \subseteq \mathbb{Z}$, res. field $\mathbb{F}_p$

$$\zeta_{\mathcal{X}}(s) = \prod_p Z_{\{pt\}/\mathbb{F}_p}(p^{-s}) = \prod_p \frac{1}{1-p^{-s}} = \text{Riemann } \zeta(s)$$

Ex $\mathcal{X} = \mathbb{A}^n$ $\qquad \zeta_{\mathcal{X}}(s) = \prod_p \frac{1}{1-p^n p^{-s}} = \zeta(s-n)$

$\mathcal{X} = \mathbb{P}^1$ $\qquad \zeta_{\mathcal{X}}(s) = \zeta(s-1)\,\zeta(s)$ $\qquad (\mathbb{A}^1 \amalg \{pt\})$

$\mathcal{X}: xy = 0 \subseteq \mathbb{A}^2$ $\qquad \zeta_{\mathcal{X}}(s) = \zeta(s-1)^2/\zeta(s)$

$\mathcal{X}: y^2 = x^3 \subseteq \mathbb{A}^2$ $\qquad \zeta_{\mathcal{X}}(s) = \zeta(s-1)$

Ex $\mathcal{X}: x^2+1 = 0 \subseteq \mathbb{A}^1_{\mathbb{Z}}$, so $\mathcal{X} = \operatorname{Spec} \frac{\mathbb{Z}[x]}{x^2+1} = \operatorname{Spec} \mathbb{Z}[i]$

closed pts = non-zero prime ideals $\mathfrak{p} \subseteq \mathbb{Z}[i]$, $|k_{\mathfrak{p}}| =: N\mathfrak{p}$.

$$\zeta_{\mathcal{X}}(s) = \prod_{\mathfrak{p}} \frac{1}{1-N\mathfrak{p}^{-s}} = \zeta_{\mathbb{Q}(i)}(s). \qquad \text{Dedekind}$$

$\mathcal{X}': x^2+9 = 0 \subseteq \mathbb{A}^1_{\mathbb{Z}}$ $\qquad X' = \operatorname{Spec} \mathbb{Z}[3i]$ ← non-maximal order; $X'$ singular scheme while $X$ is regular ($\mathbb{Z}[i]$ integrally closed).

Here $\mathcal{X}_p \cong \mathcal{X}'_p$ for all $p \neq 3$ via $x \mapsto 3x$

So

$$\zeta_{\mathcal{X}'}(s) = \zeta_{\mathcal{X}}(s) \cdot \text{some rational function of } 3^{-s}$$ ← in particular also meromorphic.

Generally:

**Thm** Suppose $X/\mathbb{Q}$ finite type of dim 0.

Let $\mathcal{X}/\mathbb{Z}$ be any model of $X$, so $\mathcal{X}_{\mathbb{Q}} \cong X$.

Then $\zeta_{\mathcal{X}}(s)$ has meromorphic cont. to $\mathbb{C}$.

**Proof**

**Step 1** As for $\mathbb{Z}[i]$ vs $\mathbb{Z}[3i]$, this is independent of the model.

**Step 2** May assume $X$ reduced; then

$$X \cong \operatorname{Spec}(K_1 \times \cdots \times K_n) \qquad K_i \text{ number fields.}$$
$$= \operatorname{Spec} \mathbb{Q}[x]/(f) \qquad f = f_1 \cdots f_n$$

**Step 3** Choose $\mathcal{X} = \mathcal{O}_{K_1} \times \cdots \times \mathcal{O}_{K_n}$

Then $\zeta_{\mathcal{X}}(s) = \prod \zeta_{K_i}(s)$ meromorphic

$\blacksquare$

**Summary**

$X/\mathbb{Q}$ of dim 0, as above

$$X \rightsquigarrow \zeta_{\mathcal{X}}(s) \text{ for some model } \mathcal{X}/\mathbb{Z} \text{ of } X/\mathbb{Q} \quad \overset{\text{up to fin. many Euler factors}}{\doteq} \quad L(V_X, s) \quad \text{Artin}$$

where, writing $X = \operatorname{Spec} \mathbb{Q}[x]/(f)$

$$V_X = \mathbb{C}[\text{roots of } f \text{ in } \overline{\mathbb{Q}}] = \mathbb{C}[X(\overline{\mathbb{Q}})]$$
$$\circlearrowleft$$
$$\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

$\leftarrow$ Note: No models $/\mathbb{Z}$ involved, purely in terms of $X/\mathbb{Q}$.
E.g. $\zeta_{K_i}(s)$ can be constructed without using $\mathcal{O}_K$!

**Goal:** Give a representation-theoretic interpretation of $\zeta_{\mathcal{X}}(s)$ for $\mathcal{X}/\mathbb{Z}$ of any dimension, and analyse it in detail for elliptic curves.

## § Étale cohomology

$M$ compact $\mathbb{C}$-manifold $\rightsquigarrow$ Betti cohomology groups

$M \rightsquigarrow H^i_{Betti}(M, \mathbb{Z})$ $\leftarrow$ fin. gen. $\mathbb{Z}$-modules, rank $= i$th Betti number, functorial in $M$, $0 \leq i \leq \dim_{\mathbb{R}} M = 2\dim_{\mathbb{C}} M$.

Ex $M$ compact Riemann surface of genus $g$  $H^0 \cong \mathbb{Z}$, $H^1 \cong \mathbb{Z}^{2g}$, $H^2 \cong \mathbb{Z}$.

$X/K$ non-singular proj. variety, $K$ any field, $\ell \neq \operatorname{char} K$ prime.

Grothendieck, Deligne, Verdier:

$$X \rightsquigarrow H^i_{ét}(X_{\bar{K}}, \mathbb{Q}_\ell) \circlearrowleft$$

étale coh. gp., $0 \leq i \leq 2\dim X$

$G_K$ acts continuously

**Thm** $\mathfrak{X}/\mathbb{Z}$ proper, flat, $p$ prime s.t. $\mathfrak{X}_{\mathbb{Q}}, \mathfrak{X}_p$ non-singular of dim $d$ ("good reduction")

$I_p < D_p < \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ inertia / decomposition gp. at $p$. Then

- $H^i_{Betti}(X_{\mathbb{Q}}(\mathbb{C}), \mathbb{Q}_\ell) \underset{\text{as } \mathbb{Q}_\ell\text{-v.spaces}}{\cong} H^i_{ét}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) \underset{\text{as } \operatorname{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)\text{-modules}}{\cong} H^i_{ét}(\mathfrak{X}_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)$.

  $\circlearrowleft$ $I_p$ acts trivially

  $\Rightarrow D_p/I_p \cong \operatorname{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$-module

- eigenvalues of $\operatorname{Frob}_p^{-1}$ on RHS have abs. value $p^{i/2}$.

- $Z_{\mathfrak{X}_{\mathbb{F}_p}}(T) = \prod_{i=0}^{d} \det(1 - \operatorname{Frob}_p^{-1} T \mid \text{RHS})^{(-1)^i}$ $\leftarrow$ $\det(\dots) \in \mathbb{Z}[T]$, independent of $\ell$

**Def** If $(V_\ell)_{\ell \text{ prime}}$ collection of $\mathbb{Q}_\ell$-vector spaces with continuous $\operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action, $S$ finite set of primes s.t.

① $p \notin S \cup \{\ell\} \Rightarrow I_p$ acts trivially on $V_\ell$ ($V_\ell$ unramified at $p$)

② $F_p(T) = \det(1 - \operatorname{Frob}_p^{-1} T \mid V_\ell^{I_p})$ is in $\mathbb{Q}[T]$ and independent of $\ell$ for all $p \neq \ell$

We call $(V_\ell)_\ell$ a <u>strictly compatible system of $\ell$-adic reps</u>

If ② only holds for $p \notin S$, we call it a <u>compatible system</u>.