Def For $V = (V_\ell)_{\ell \text{ prime}}$ strictly compatible,

$$L(V,s) := \prod_p F_p(p^{-s})^{-1} \qquad \underline{\text{L-function of } V}$$

~~ Lecture 7 ~~

Ex Artin rep.

can replace $(\mathbb{Q}_\ell)_\ell$ by $(F_\lambda)_{\lambda \text{ prime of } F}$, for F number field to include all Artin reps, e.g. Dirichlet chars. "λ-adic system".

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \underset{\text{finite}}{\mathrm{Gal}(K/\mathbb{Q})} \longrightarrow GL_d(\overline{\mathbb{Q}}) \hookrightarrow GL_d(\overline{\mathbb{Q}}_\ell) \quad \forall \ell$$

strictly compatible, $S = \{\text{primes ramified in } K/\mathbb{Q}\}$

Ex $X/\mathbb{Q}$ non-sing. proj. variety, $0 \le i \le 2d$, $d = \dim X$.

$V_\ell^{(i)} = H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ compatible by Thm., $S = \{\text{bad primes for } X/\mathbb{Q}\}$,

strictly compatible when $i = 0, 1, 2d-1, 2d$, conjecturally always ← not known!

By Thm., for any model $\mathcal{X}/\mathbb{Z}$ of $X/\mathbb{Q}$,

$$\zeta_{\mathcal{X}}(s) \doteq \prod_{i=0}^{2d} L(V_\ell^{(i)}, s)^{(-1)^i} \qquad \forall \ell.$$

up to fin. many Euler factors

rep-theoretic description of $\zeta_{\mathcal{X}}$;

Conj These have fun. eq. $s \leftrightarrow i+1-s$ and Riemann hypothesis.

Rmk

- $(V_\ell)_{\ell \text{ prime}}$ = "poor man's version" of one rep. $V/\mathbb{Q}$
- Have standard constructions $\oplus$, $\otimes$, Ind, Res for (strictly) compatible systems, conductors etc. and Artin formalism.
- In practice, all come from H^i_{et}, modular forms, Artin reps and $\otimes$ ("twisted L-functions").
- 1-dim compatible systems all come from Hecke characters.
- Langlands: all compatible systems should come from automorphic reps.

§ H^i_{et} that we know $(0, 1, 2d-1, 2d)$

There is 1-dim rep. $\chi_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}\, T_\ell \mathbb{G}_m \cong \mathbb{Z}_\ell^\times \subseteq GL_1(\mathbb{Q}_\ell)$

action on ℓ-power roots of unity in $\overline{\mathbb{Q}}$; I_p acts trivially, Frob_p acts as p.

called ℓ-adic cyclotomic character, also denoted $\mathbb{Q}_\ell(1)$.

a strictly compatible system,

$L(\mathbb{Q}_\ell(1), s) = \zeta(s+1)$.

$$\underbrace{L(\mathbb{Q}_\ell(1)^{\otimes n}}_{=:\ \mathbb{Q}_\ell(n)}, s) = \zeta(s+n) \quad , n \in \mathbb{Z}$$

generally for any $(V_\ell)_\ell$

$L(V_\ell \otimes \mathbb{Q}_\ell(n)) = L(V_\ell, s+n)$ ← allows shifts by $n \in \mathbb{Z}$; convenient

called Tate twist

Thm X/K complete variety, K perfect, $\ell \neq \mathrm{char}\, K$, $d = \dim X$. Then

- $H^0_{et}(X_{\overline{K}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell[\text{connected components of } X_{\overline{K}}]$.

  $\circlearrowleft$ $\mathrm{Gal}(\overline{K}/K)$

- $H^{2d}_{et}(X_{\overline{K}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell[\text{irreducible components of } X_{\overline{K}}] \otimes \mathbb{Q}_\ell(-d)$

Assume X smooth, integral:

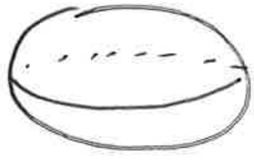
- $H^1_{et}(X_{\overline{K}}, \mathbb{Q}_\ell) = \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell \mathrm{Pic}\, X, \mathbb{Q}_\ell) = V_\ell \mathrm{Pic}\, X^*$

- $H^{2d-n}_{et}(X_{\overline{K}}, \mathbb{Q}_\ell) \cong H^n_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)^* \otimes \mathbb{Q}_\ell(-d)$.

Ex $\dim X = 0$ $\rightsquigarrow$ only have H^0_{et}

$$H^0_{et}(X_{\bar{K}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell[\text{conn. comps of } \underset{\text{points.}}{X_{\bar{K}}}] = \mathbb{Q}_\ell[X(\bar{K})]$$

Ex $X = \mathbb{P}^1_{\mathbb{Q}}$



$\mathbb{P}^1(\mathbb{C})$ Riemann sphere
Betti numbers 1, 0, 1

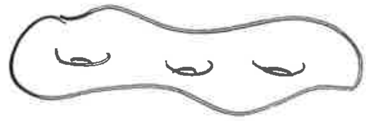
$H^0 \cong \mathbb{Q}_\ell$

$H^1 = 0$

$H^2 \cong \mathbb{Q}_\ell(-1)$

$$\zeta_{\mathbb{P}^1/\mathbb{Z}}(s) = \frac{\overset{H^0}{\zeta(s)}\ \overset{H^2}{\zeta(s-1)}}{\underset{H^1}{1}}$$

Ex $X/\mathbb{Q}$ non-singular proj. curve, genus g.



Betti numbers $1, 2g, 1$

$H^0 \cong \mathbb{Q}_\ell$ $\rightsquigarrow \zeta(s)$

$H^1 \cong V_\ell \operatorname{Jac} C^*$ $\rightsquigarrow L(C,s)$

$H^2 \cong \mathbb{Q}_\ell(1)$ $\rightsquigarrow \zeta(s-1)$

Def $L(X,s) = L(H^1_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell), s) = \prod_p F_p(p^{-s})^{-1}$

$F_p(T) = \det(1 - \mathrm{Frob}_p^{-1} T \mid (V_\ell \operatorname{Jac} C^*)^{I_p})$

known to be independent of ℓ.

L-function of a curve; of degree $2g$

Conj $L(X,s)$ entire, satisfies fun. eq. $s \leftrightarrow 2-s$.

[+ RH
+ BSD, ...]

Known for ell. curves

$/\mathbb{Q}$ Taylor-Wiles, BCDT.

$/\mathbb{Q}(\sqrt{d})$, $d>0$ Freitas-LeHung-Siksek

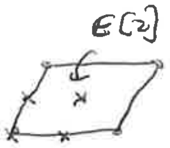
$/\mathbb{Q}(\sqrt{d})$, $d<0$ often Caraiani-Newton

/many CM fields (ACCGHHNSTT)

/all tot. real fields meromorphic + fun. eq. (Taylor, uses Brauer induction).

Goal: Understand $L(E,s)$ for ell. curve E in terms of geometry of $E/\mathbb{Q}_p$.

§ Tate module of an elliptic curve



$E/\mathbb{Q}$ ell. curve ; $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$

$$\underset{\substack{\nwarrow \\ \ell^n \text{ torsion}}}{E[\ell^n]} \cong \Lambda / \tfrac{1}{\ell^n}\Lambda \cong \underset{\substack{\circlearrowleft \\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}}{(\mathbb{Z}/\ell^n\mathbb{Z})^2} \quad \text{as ab. gps}$$

<u>Def</u> $T_\ell E = \varprojlim_n E[\ell^n] \cong \mathbb{Z}_\ell^2$

$V_\ell E = T_\ell E \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong \mathbb{Q}_\ell^2$ $\qquad \circlearrowleft \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on all.

$H^1_{et}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = V_\ell E^* \cong \mathbb{Q}_\ell^2$

Want to understand how D_p and I_p act on H^1

$$\begin{array}{ccccc} \overline{\mathbb{Q}} & \mathfrak{P} & \text{complete} & \overline{\mathbb{Q}_p} \\ | & | & \rightsquigarrow & | \\ \mathbb{Q} & p & & \mathbb{Q}_p \end{array}$$

<u>Fact</u>
- $D_p \cong \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$
- primes $\mathfrak{P} | p$ in $\overline{\mathbb{Q}}$ $\overset{1:1}{\longleftrightarrow}$ embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$.

Need to understand $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \circlearrowright T_\ell E_{\mathbb{Q}_p}$, $\leftarrow$ purely local question

relate it to geometry of $E/\mathbb{Q}_p$, and compute

$$\boxed{F(T) = \det(1 - \mathrm{Frob}\cdot T \,|\, (V_\ell E_p^*)^I)}$$

Recall:

$$\begin{array}{lll} & \overline{\mathbb{Q}_p} & \\ & \quad | \ I_{wild} & \\ I & \bigcup_{p\nmid n} \mathbb{Q}_p(\zeta_n, \sqrt[n]{p}) & \text{max tame ext. } \mathbb{Q}_p^{tame} \\ & \quad | \ I_{tame} & \\ & \bigcup_{p\nmid n} \mathbb{Q}_p(\zeta_n) & \text{max unr. ext. } \mathbb{Q}_p^{nr} \\ \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong & \quad | & \\ & \mathbb{Q}_p & \end{array}$$

gen. by Frob: $x \mapsto x^p$

Ex Look at $T_\ell E_i$ for $\ell = 2$ for :

$E_1/\mathbb{Q}_5 : \; y^2 = x^3 - 1$

$E_2/\mathbb{Q}_5 : \; y^2 = (x-1)(x^2-5)$

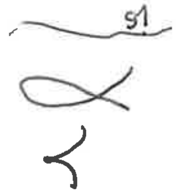
$E_3/\mathbb{Q}_5 : \; y^2 = x^3 - 5$

reduce mod 5 $\rightsquigarrow$

$\tilde{E}_1/\mathbb{F}_p$ good reduction

$\tilde{E}_2/\mathbb{F}_5$ mult. reduction

$\tilde{E}_3/\mathbb{F}_5$ additive reduction



Compute $E_i[2^n]$ for $n = 1,2,3$ (Magma)

$E_1[2] = \{0, (1,0), (\zeta_3, 0), (\zeta_3^2, 0)\}$ unramified (all coordinates $\in \mathbb{Q}_5^{nr}$
"all visible in $\overline{\mathbb{F}_5}$" $\iff I$ acts trivially)

$E_2[2]$, $E_3[2]$ tamely ramified (all coords $\in \mathbb{Q}_5^{tame}$ $\iff I_{wild}$ trivially)

	E_1:	E_2:	E_3:
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbb{Q}_5(E[8])$	$\mathbb{Q}_5(\zeta_{16})$	$\mathbb{Q}_5(\sqrt[4]{-5}, \zeta_8)$	$\mathbb{Q}_5(\zeta_{16}, \sqrt[3]{5})$
$\mid$	$1 \mid$	$\mid$	$\mid$
$\mathbb{Q}_5(E[4])$	$\mathbb{Q}_5(\zeta_{16})$	$\mathbb{Q}_5(\sqrt[4]{-5})$	$\mathbb{Q}_5(\zeta_{16}, \sqrt[3]{5})$
$\mid$	$2 \mid$	$\mid$	$\mid$
$\mathbb{Q}_5(E[2])$	$\mathbb{Q}_5(\zeta_8)$	$\mathbb{Q}_5(\sqrt{5})$	$\mathbb{Q}_5(\zeta_8, \sqrt[3]{5})$
$\mid$	$2 \mid$	$\mid$	$\mid$
$\mathbb{Q}_5$	$\mathbb{Q}_5$	$\mathbb{Q}_5$	$\mathbb{Q}_5$

(Note: $\mathbb{Q}_5(\zeta_8) = \mathbb{Q}_5(\zeta_{24}) = \mathbb{Q}_5(\zeta_3) = \mathbb{Q}_5(\sqrt{-3})$ quad unr. ext.)

Residue degree : Weil pairing $E[2^n] \times E[2^n] \longrightarrow$ 2^n th roots of unity $\Rightarrow \zeta_{2^n} \in \mathbb{Q}_5(E[2^n])$

$\Rightarrow$ res. degree $/\mathbb{Q}_5 \to \infty$ as $n \to \infty$.

Ramification: Guess? E_1 all unramified

E_2 regularly growing ramification

E_3 finite ramification

[Yes! For any $\ell \neq 5$.
$\iff$ good red.
$\iff$ pot. mult. red.
$\iff$ additive pot. good. red]

§ Good reduction

Thm (Néron-Ogg-Shafarevich) $E/\mathbb{Q}_p$ ell. curve, $\ell \neq p$.

$E/\mathbb{Q}_p$ has good red. $\iff$ $E[\ell^n]$ unramified $\forall n \geq 1$ $\iff$ $T_\ell E$ unramified

$\iff$ $V_\ell E$ unramified

In this case, $\mathrm{Frob} \circlearrowright T_\ell E$ as a matrix with char. poly

$$T^2 - aT + p \qquad\qquad a = p+1-\#\tilde{E}(\mathbb{F}_p)$$

and so does $\underbrace{\mathrm{Frob}^{-1}}_{\text{invert eigenvalues}}$ on $\underbrace{(V_\ell E)^*}_{\text{invert again.}}$

$$\boxed{F(T) = \det(1-\mathrm{Frob}^{-1}\cdot T \mid (V_\ell E^*)^I) = 1 - aT + pT^2}$$

Ex $E = E_1/\mathbb{Q}_5$: $y^2 = x^3 - 1$ good red.

$\tilde{E}_1/\mathbb{F}_5$: $y^2 = x^3-1$ has 6 pts $\{O, (0,\pm 2), (1,0), (3,\pm 2)\}$

$$T^2 - aT + q = T^2 + 5 \quad \text{and} \quad F(T) = \underline{1 + 5T^2}$$

Here $I \circlearrowright T_2 E$ as $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and Frob as $\begin{pmatrix}0&-5\\1&0\end{pmatrix}$ in some basis.

So $\mathbb{Q}_5(E[2^n])/\mathbb{Q}_5$ unramified of degree

$$f = \text{order of } \begin{pmatrix}0&-5\\1&0\end{pmatrix} \in GL_2(\mathbb{Z}/2^n\mathbb{Z}) = \overbrace{2,4,4}^{\text{last page}},8,16,\ldots$$

for $n = 1,2,3,4,5,\ldots$

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lecture 8

§ Multiplicative reduction.

For $E/\mathbb{C}$

$$E(\mathbb{C}) \cong \mathbb{C}/_{\mathbb{Z}+\tau\mathbb{Z}} \xrightarrow[\exp(2\pi i\,\cdot)]{\sim} \mathbb{C}^*/_{q^{\mathbb{Z}}} \qquad q = e^{2\pi i \tau}$$

[$\cong$ given by transcendental fncs]

as complex Lie groups.