Diophantine Geometry, Fundamental Groups, and Non-Abelian Reciprocity

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Figure: John Coates at 700

Diophantine Geometry: Abelian Case

The *Hasse-Minkowski theorem* says that

$$ax^2 + by^2 = c$$

has a solution in a number field F and only if it has a solution in F_v for all v.

There are straightforward algorithms for determining this. For example, we need only check for $v = \infty$ and $v|2abc$, and there, a solution exists if and only if

$$(a, b)_v(b, c)_v(c, a)_v(c, -1)_v = 1.$$

Diophantine Geometry: Main Local-to-Global Problem

Locate

$$X(F) \subset X(\mathbb{A}_F) = \prod_v{}' X(F_v)$$

The question is

How do the global points sit inside the local points?

In fact, there is a classical answer of satisfactory sort for conic equations.

Diophantine Geometry: Main Local-to-Global Problem

In that case, assume for simplicity that there is a rational point (and that the points at infinity are rational), so that

$$X \simeq \mathbb{G}_m.$$

Then

$$X(F) = F^*, \quad X(F_v) = F_v^*.$$

Problem becomes that of locating

$$F^* \subset \mathbb{A}_F^\times.$$

Diophantine Geometry: Abelian Class Field Theory

We have the Artin reciprocity map

$$\mathrm{Rec} = \prod_v \mathrm{Rec}_v : \mathbb{A}_F^\times \longrightarrow G_F^{ab}.$$

Here,

$$G_F^{ab} = \mathrm{Gal}(F^{ab}/F),$$

and

$$F^{ab}$$

is the maximal abelian algebraic extension of F.

Diophantine Geometry: Abelian Class Field Theory

Artin's reciprocity law:

The map

$$F^* \hookrightarrow \mathbb{A}_F^\times \xrightarrow{\text{Rec}} G_F^{ab}$$

is zero.

That is, the reciprocity map gives a *defining equation for* $\mathbb{G}_m(F)$.

Diophantine Geometry: Non-Abelian Reciprocity?

We would like to generalize this to other equations by way of a *non-abelian reciprocity law*.

Start with a rather general variety X for which we would like to understand

$$X(F)$$

via

$$X(F) \hookrightarrow X(\mathbb{A}_F) \xrightarrow{\mathrm{Rec}^{NA}} \boxed{\text{some target with base-point } 0}$$

in such way that

$$\mathrm{Rec}^{NA} = 0$$

becomes an equation for $X(F)$.

Diophantine Geometry: Non-Abelian Reciprocity

To rephrase: we would like to construct *class field theory with coefficients in a general variety X* generalizing CFT with coefficients in $\mathbb{G}_m$

Will describe a version that works for smooth hyperbolic curves.

Diophantine Geometry: Non-Abelian Reciprocity

(Joint with Jonathan Pridham)

Notation:

F: number field.
$G_F = \mathrm{Gal}(\bar{F}/F)$.
$G_v = \mathrm{Gal}(\bar{F}_v/F_v)$ for a place v of F.
S: finite set of places of F.
$\mathbb{A}_F$: Adeles of F
$\mathbb{A}_F^S$: S-integral adeles of F.
$G_F^S = \mathrm{Gal}(F^S/F)$, where F^S is the maximal extension of F unramified outside S.

Diophantine Geometry: Non-Abelian Reciprocity

X: a smooth curve over F with genus at least two; $b \in X(F)$ (sometimes tangential).

$$\Delta = \pi_1(\bar{X}, b):$$

Pro-finite étale fundamental group of $\bar{X} = X \times_{\mathrm{Spec}(F)} \mathrm{Spec}(\bar{F})$ with base-point b.

$$\Delta^{[n]}$$

Lower central series with $\Delta^{[1]} = \Delta$.

$$\Delta_n = \Delta/\Delta^{[n+1]}.$$

$$T_n = \Delta^{[n]}/\Delta^{[n+1]}.$$

Diophantine Geometry: Non-Abelian Reciprocity

We then have a *nilpotent class field theory with coefficients in* X made up of a filtration

$$X(\mathbb{A}_F) = X(\mathbb{A}_F)_1 \supset X(\mathbb{A}_F)_2 \supset X(\mathbb{A}_F)_3 \supset \cdots$$

and a sequence of maps

$$Rec_n : X(\mathbb{A}_F)_n \longrightarrow \mathfrak{G}_n(X)$$

to a sequence $\mathfrak{G}_n(X)$ of profinite abelian groups in such a way that

$$X(\mathbb{A}_F)_{n+1} = Rec_n^{-1}(0).$$

Diophantine Geometry: Non-Abelian Reciprocity

$$\begin{array}{ccccccc}
\cdots & \subset & X(\mathbb{A}_F)_3 = Rec_2^{-1}(0) & \subset & X(\mathbb{A}_F)_2 = Rec_1^{-1}(0) & \subset & X(\mathbb{A}_F)_1 = X(\mathbb{A}_F) \\
\cdots & & \downarrow Rec_3 & & \downarrow Rec_2 & & \downarrow Rec_1 \\
\cdots & & \mathfrak{G}_3(X) & & \mathfrak{G}_2(X) & & \mathfrak{G}_1(X)
\end{array}$$

Rec_n is defined not on all of $X(\mathbb{A}_F)$, but only on the kernel (the inverse image of 0) of all the previous rec_i.

Diophantine Geometry: Non-Abelian Reciprocity

The $\mathfrak{G}_n(X)$ are defined as

$$\mathfrak{G}_n(X) :=$$

$$\mathrm{Hom}[H^1(G_F, D(T_n)), \mathbb{Q}/\mathbb{Z}]$$

where

$$D(T_n) = \varinjlim_m \mathrm{Hom}(T_n, \mu_m).$$

When $X = \mathbb{G}_m$, then $\mathfrak{G}_n(X) = 0$ for $n \geq 2$ and

$$\mathfrak{G}_1 = \mathrm{Hom}[H^1(G_F, D(\hat{\mathbb{Z}}(1))), \mathbb{Q}/\mathbb{Z}]$$

$$= \mathrm{Hom}[H^1(G_F, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}] = G_F^{ab}.$$

Diophantine Geometry: Non-Abelian Reciprocity

The reciprocity maps are defined using the local period maps

$$j^v : X(F_v) \longrightarrow H^1(G_v, \Delta);$$

$$x \mapsto [\pi_1(\bar{X}; b, x)].$$

Because the homotopy classes of étale paths

$$\pi_1(\bar{X}; b, x)$$

form a torsor for Δ with compatible action of G_v, we get a corresponding class in non-abelian cohomology of G_v with coefficients in Δ.

Diophantine Geometry: Non-Abelian Reciprocity

These assemble to a map

$$j^{loc} : X(\mathbb{A}_F) \longrightarrow \prod H^1(G_v, \Delta),$$

which comes in levels

$$j_n^{loc} : X(\mathbb{A}_F) \longrightarrow \prod H^1(G_v, \Delta_n).$$

Diophantine Geometry: Non-Abelian Reciprocity

The first reciprocity map is just defined using

$$x \in X(\mathbb{A}_F) \mapsto d_1(j_1^{loc}(x)),$$

where

$$d_1 : \prod^S H^1(G_v, \Delta_1^M) \longrightarrow \prod^S H^1(G_v, D(\Delta_1^M))^\vee \xrightarrow{\text{loc}^*} H^1(G_F^S, D(\Delta_1^M))^\vee,$$

is obtained from Tate duality and the dual of localization. One needs first to work with a pro-M quotient for a finite set of primes M and $S \supset M$. Then take a limit over S and then M.

Diophantine Geometry: Non-Abelian Reciprocity

To define the higher reciprocity maps, we use the exact sequences

$$0 \longrightarrow H^1_c(G^S_F, T^M_{n+1}) \longrightarrow H^1_z(G^S_F, \Delta^M_{n+1}) \longrightarrow H^1_z(G^S_F, \Delta_n)$$

$$\xrightarrow{\delta_{n+1}} H^2_c(G^S_F, T^M_{n+1})$$

for non-abelian cohomology with support and Poitou-Tate duality

$$d_{n+1} : H^2_c(G^S_F, T^M_{n+1}) \simeq H^1(G^S_F, D(T^M_{n+1}))^\vee.$$

Diophantine Geometry: Non-Abelian Reciprocity

Essentially,

$$Rec^M_{n+1} = d_{n+1} \circ \delta_{n+1} \circ \mathrm{loc}^{-1} \circ j_n.$$

$$x \in X(\mathbb{A}_F)_{n+1} \xrightarrow{j_n^{loc}} \prod^S H^1(G_v, \Delta_n^M) \xrightarrow{\mathrm{loc}^{-1}} H^1_{j_n^{loc}(x)}(G_F^S, \Delta_n^M)$$

$$\xrightarrow{\delta_{n+1}} H^2_c(G_F^S, T^M_{n+1}) \xrightarrow{d_{n+1}} H^1(G_F^S, D(T^M_{n+1}))^\vee.$$

At each stage, take a limit over S and M to get the reciprocity maps.

Diophantine Geometry: Non-Abelian Reciprocity

Put

$$X(\mathbb{A}_F)_\infty = \cap_{n=1}^{\infty} X(\mathbb{A}_F)_n.$$

Theorem (Non-abelian reciprocity)

$$X(F) \subset X(\mathbb{A}_F)_\infty.$$

Diophantine Geometry: Non-Abelian Reciprocity

Remark: When $F = \mathbb{Q}$ and p is a prime of good reduction, suppose there is a finite set T of places such that

$$H^1(G_F^S, \Delta_n^p) \longrightarrow \prod_{v \in T} H^1(G_v, \Delta_n^p)$$

is injective. Then the reciprocity law implies finiteness of $X(F)$.

Non-Abelian Reciprocity: idea of proof

$$
\begin{array}{ccc}
X(F) & \longrightarrow & X(A_F) \\
\Big\downarrow j_n^{g} & & \Big\downarrow j_n^{loc} \\
H^1(G_F^S, \Delta_n^M) & \xrightarrow{\text{loc}} & \prod H^1(G_v, \Delta_n^M)
\end{array}
$$

$$
\begin{array}{ccc}
 & & H^1(G_F^S, \Delta_{n+1}^M) \\
 & \overset{j_{n+1}^{g}}{\nearrow} & \Big\downarrow \\
X(F) & \xrightarrow{j_n^{g}} & H^1(G_F^S, \Delta_{n+1}^M)
\end{array}
$$

Non-Abelian Reciprocity: idea of proof

If $x \in X(\mathbb{A}_F)$ comes from a global point $x^g \in X(F)$, then there will be a class

$$j_n^g(x^g) \in H^1_{j_n(x)}(G_F^S, \Delta_n^M)$$

for every n corresponding to the global torsor

$$\pi_1^{et,M}(\bar{X}; b, x^g).$$

That is, $j_n^g(x^g) = \mathrm{loc}^{-1}(j_n^{loc}(x))$ and

$$\delta_{n+1}(j_n^g(x^g)) = 0$$

for every n.

A non-abelian conjecture of Birch and Swinnerton-Dyer type

Let

$$Pr_v : X(\mathbb{A}_F) \longrightarrow X(F_v)$$

be the projection to the v-adic component of the adeles.

Define

$$X(F_v)_n := Pr_v(X(\mathbb{A}_F)_n).$$

Thus,

$$X(F_v) = X(F_v)_1 \supset X(F_v)_2 \supset X(F_v)_3 \supset \cdots \supset X(F_v)_\infty \supset X(F).$$

Conjecture: Let $X/\mathbb{Q}$ be a projective smooth curve of genus at least 2. Then for any prime p of good reduction, we have

$$X(\mathbb{Q}_p)_\infty = X(\mathbb{Q}).$$

A non-abelian conjecture of Birch and Swinnerton-Dyer type

Can consider more generally integral points on affine hyperbolic X as well.

Conjecture: Let X be an affine smooth curve with non-abelian fundamental group and S a finite set of primes. Then for any prime $p \notin S$ of good reduction, we have

$$X(\mathbb{Z}[1/S]) = X(\mathbb{Z}_p)_\infty.$$

Should allow us to compute

$$X(\mathbb{Q}) \subset X(\mathbb{Q}_p)$$

or

$$X(\mathbb{Z}[1/S]) \subset X(\mathbb{Z}_p).$$

A non-abelian conjecture of Birch and Swinnerton-Dyer type

Whenever we have an element

$$k_n \in H^1(G_T, \mathrm{Hom}(T_n^M, \mathbb{Q}_p(1))),$$

we get a function

$$X(\mathbb{A}_{\mathbb{Q}})_n \xrightarrow{rec_n} H^1(G_T, D(T_n^M))^\vee \xrightarrow{k_n} \mathbb{Q}_p$$

that kills $X(\mathbb{Q}) \subset X(\mathbb{A}_{\mathbb{Q}})_n$.

Need an *explicit reciprocity law* that describes the image

$$X(\mathbb{Q}_p)_n.$$

A non-abelian conjecture of Birch and Swinnerton-Dyer type

Computational approaches all rely on the theory of

$$U(X, b),$$

the $\mathbb{Q}_p$-pro-unipotent fundamental group of $\bar{X}$ with Galois action, and the diagram

$$\begin{array}{ccccc}
X(\mathbb{Q}) & \longrightarrow & X(\mathbb{Q}_p) & & \\
\Big\downarrow{\scriptstyle j_n^g} & & \Big\downarrow{\scriptstyle j_n^p} & \searrow^{j_n^{DR}} & \\
H_f^1(G_{\mathbb{Q}}^T, U_n) & \xrightarrow{loc_n^p} & H_f^1(G_p, U_n) & \xrightarrow[\simeq]{D} & U_n^{DR}/F^0
\end{array}$$

A non-abelian conjecture of Birch and Swinnerton-Dyer type

The key point is that the map

$$X(\mathbb{Q}_p) \xrightarrow{j^{DR}} U^{DR}/F^0$$

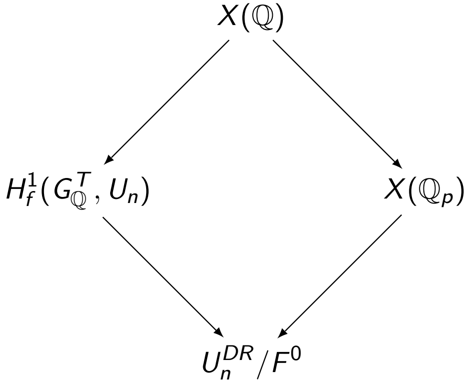
can be computed explicitly using iterated integrals, and

$$X(\mathbb{Q}) \subset X(\mathbb{Q}_p)_n \subset [j_n^{DR}]^{-1}[Im(D \circ \mathrm{loc}_n^p)].$$

A non-abelian conjecture of Birch and Swinnerton-Dyer type

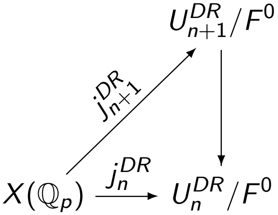
Two more key facts:

1. As soon as $D \circ \mathrm{loc}_n^p$ has non-dense image, $X(\mathbb{Q}_p)_n$ is finite. This follows from analytic properties of Coleman functions and the fact that j_n^{DR} has dense image. That is, in this case, $Im(j_n^{DR}) \cap Im(D \circ \mathrm{loc}_p)$ is finite.



A non-abelian conjecture of Birch and Swinnerton-Dyer type

2. If $\mathcal{A}_n^{DR}$ denotes the coordinate ring of U_n^{DR}/F^0, then the functions $[j_{n+1}^{DR}]^*(\mathcal{A}_{n+1}^{DR})$ contains many elements algebraically independent from $[j_n^{DR}]^*(\mathcal{A}_n^{DR})$.



A non-abelian conjecture of Birch and Swinnerton-Dyer type

Predicted phenomena: At some point $X(\mathbb{Q}_p)_n$ should be finite, and then one should have a strongly increasing set of functions

$$[J_m^{DR}]^*(I_m^{DR})$$

for $m \geq n$ that vanish on $X(\mathbb{Q})$.

This is implied, for example, by the Fontaine-Mazur conjecture on geometric Galois representations, which implies

$$\dim[U_n^{DR}/F^0] - \dim[Im(D \circ \mathrm{loc}_n^p)] \longrightarrow \infty$$

as n grows.

Can prove this for curves X that have CM Jacobians (joint with J. Coates).

A non-abelian conjecture of Birch and Swinnerton-Dyer type: Examples [Joint with Jennifer Balakrishnan, Ishai Dan-Cohen, Stefan Wewers]

Let $\mathcal{X} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Then $\mathcal{X}(\mathbb{Z}) = \phi$.

$$\mathcal{X}(\mathbb{Z}_p)_2 = \{z \mid \log(z) = 0, \log(1-z) = 0\}.$$

Must have $z = \zeta_n$ and $1 - z = \zeta_m$, and hence, $z = \zeta_6$ or $z = \zeta_6^{-1}$.

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples

Thus, if $p = 3$ or $p \equiv 2 \mod 3$, we have

$$\mathcal{X}(\mathbb{Z}_p)_2 = \phi = \mathcal{X}(\mathbb{Z}),$$

so the conjecture holds already at level 2.

When $p \equiv 1 \mod 3$

$$\mathcal{X}(\mathbb{Z}) = \phi \subsetneq \{\zeta_6, \zeta_6^{-1}\} = \mathcal{X}(\mathbb{Z}_p)_2$$

and we must go to a higher level.

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples

Let

$$Li_2(z) = \sum_n \frac{z^n}{n^2}$$

be the *dilogarithm*. Then

$$\mathcal{X}(\mathbb{Z}_p)_3 = \{z \mid \log(z) = 0, \log(1-z) = 0, Li_2(z) = 0\}.$$

and the conjecture is true for $\mathcal{X}(\mathbb{Z})$ if

$$Li_2(\zeta_6) \neq 0.$$

Can check this numerically for all $2 < p < 10^5$.

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples

Let $\mathcal{X} = \mathcal{E} \setminus O$ where $\mathcal{E}$ is a semi-stable elliptic curve of rank 0 and $|\text{Ш}(E)(p)| < \infty$.

$$\log(z) = \int_b^z (dx/y).$$

(b is a tangential base-point.)

Then

$$\mathcal{X}(\mathbb{Z}_p)_2 = \{z \in X(\mathbb{Z}_p) \mid \log(z) = 0\} = \mathcal{E}(\mathbb{Z}_p)[tor] \setminus O.$$

For small p, it happens frequently that

$$\mathcal{E}(\mathbb{Z})[tor] = \mathcal{E}(\mathbb{Z}_p)[tor]$$

and hence that

$$\mathcal{X}(\mathbb{Z}) = \mathcal{X}(\mathbb{Z}_p)_2.$$

But of course, this fails as p grows.

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples

Must then examine the inclusion

$$\mathcal{X}(\mathbb{Z}) \subset \mathcal{X}(\mathbb{Z}_p)_3.$$

Let

$$D_2(z) = \int_b^z (dx/y)(xdx/y).$$

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples

Let S be the set of primes of bad reduction. For each $l \in S$, let

$$N_l = \mathrm{ord}_l(\Delta_{\mathcal{E}}),$$

where $\Delta_{\mathcal{E}}$ is the minimal discriminant.
Define a set

$$W_l := \{(n(N_l - n)/2N_l) \log l \mid 0 \leq n < N_l\},$$

and for each $w = (w_l)_{l \in S} \in W := \prod_{l \in S} W_l$, define

$$\|w\| = \sum_{l \in S} w_l.$$

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples

Theorem

Suppose $\mathcal{E}$ *has rank zero and that* $Ш_E[p^\infty] < \infty$. *With assumptions as above*

$$\mathcal{X}(\mathbb{Z}_p)_3 = \cup_{w \in W} \Psi(w),$$

where

$$\Psi(w) := \{z \in \mathcal{X}(\mathbb{Z}_p) \mid \log(z) = 0,\ D_2(z) = \|w\|\}.$$

Of course,

$$\mathcal{X}(\mathbb{Z}) \subset \mathcal{X}(\mathbb{Z}_p)_3,$$

but depending on the reduction of $\mathcal{E}$, the latter could be made up of a large number of $\Psi(w)$, creating potential for some discrepancy.

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples

The curve

$$y^2 + xy = x^3 - x^2 - 1062x + 13590$$

has integral points

$$(19, -9), \quad (19, -10).$$

We find

$$\mathcal{X}(\mathbb{Z}) = \{z \mid \log(z) = 0, D_2(z) = 0\} = \mathcal{X}(\mathbb{Z}_p)_3$$

for all p such that $5 \leq p \leq 97$.

Note that

$$D_2(19, -9) = D_2(19, -10) = 0$$

is already non-obvious. (A non-abelian reciprocity law.)

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples

In fact, so far, we have checked

$$\mathcal{X}(\mathbb{Z}) = \mathcal{X}(\mathbb{Z}_p)_3$$

for the prime $p = 5$ and 256 semi-stable elliptic curves of rank zero.

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples

Cremona label	number of $\|\|w\|\|$-values
1122m1	128
1122m2	384
1122m4	84
1254a2	140
1302d2	96
1506a2	112
1806h1	120
2442h1	78
2442h2	84
2706d2	120
2982j1	160
2982j2	140
3054b1	108

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples

Cremona label	number of $\|\|w\|\|$-values
3774f1	120
4026g1	90
4134b1	90
4182h1	300
4218b1	96
4278j1	90
4278j2	100
4434c1	210
4774e1	224
4774e2	192
4774e3	264
4774e4	308
4862d1	216

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples

Hence, for example, for the curve $1122m2$,

$$y^2 + xy = x^3 - 41608x - 90515392$$

there are potentially 384 of the $\Psi(w)$'s that make up $\mathcal{X}(\mathbb{Z}_p)_3$.

Of these, all but 4 end up being empty, while the points in those $\Psi(w)$ consist exactly of the integral points

$$(752, -17800), (752, 17048), (2864, -154024), (2864, 151160).$$

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples [Netan Dogra and Jennifer Balakrishnan]

$$X : y^2 = x^6 - 4x^4 + 3x^2 + 1;$$

$$E_1 : y^2 = x^3 - 4x^2 + 3x + 1;$$

$$E_2 : y^2 = x^3 + 3x^2 - 4x + 1;$$

$$f_1 : X \longrightarrow E_1;$$

$$(x, y) \mapsto (x^2, y);$$

$$f_2 : X \longrightarrow E_2;$$

$$(x, y) \mapsto (1/x^2, y/x^3);$$

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples [Netan Dogra and Jennifer Balakrishnan]

$z_1 \in E_1(\mathbb{Q}), z_2 \in E_2(\mathbb{Q})$, generators for Mordell-Weil group.

h_i, p-adic height on $E_i(\mathbb{Q})$.

$\log_i$, p-adic log on $E_i(\mathbb{Q}_p)$ with respect to suitable choice of invariant differential form.

λ_i, local p-adic height on $E_i(\mathbb{Q}_p)$. Hence, given by log of p-adic sigma function.

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples [Netan Dogra and Jennifer Balakrishnan]

Define $\rho : X(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$ by

$$\rho(z)$$

$$= 2\lambda_1(f_1(z)) - 2\frac{\log_1^2(f_1(z))}{\log_1^2(z_1)} h_1(z_1) - \lambda_2(f_2(z) - (0,1)) - \lambda_2(f_2(z) + (0,1))$$

$$+ \frac{\log_2^2(f_2(z) - (0,1)) + \log_2^2(f_2(z) + (0,1)))}{\log_1^2(z_2)} h_2(z_2).$$

Then

$$X(\mathbb{Q}_p)_3 \subset \{\rho(z) = \log 2\} \cup \{\rho(z) = 2\log 2\} \cup \{\rho(z) = (-1/3)\log 2\}$$

A non-abelian conjecture of Birch and Swinnerton-Dyer type: examples [Netan Dogra and Jennifer Balakrishnan]

Get some nice explicit reciprocity laws like

$$\rho(0, \pm 1) = \log 2;$$

$$\rho(5/2, \pm 83/8) = 2 \log 2;$$

$$\rho(1, \pm 1) = (-1/3) \log 2.$$

Non-abelian reciprocity: a brief comparison

Usual (Langlands) reciprocity:

$$L(M) = L(\pi)$$

where M is a motive and π is an algebraic automorphic representation on $GL_n(\mathbb{A}_F)$.

The relevance to arithemic comes from conjectures that say $L(N^* \otimes M)$ encodes

$$RHom(N, M).$$

So in some sense, L functions classify motives.

However, in classical (non-linear) Diophantine geometry, we are interested in schemes, not motives, in particular, actual maps between schemes. Hence, a need for a nonlinear reciprocity of some sort.

Non-abelian reciprocity: a brief comparison

X/F as above, Δ_n, $T_n = \Delta^n/\Delta^{n+1}$, etc.

Langlands reciprocity

$\rho \in H^1(G_F, GL(T_1)) \mapsto$ functions on $GL(H_1^{DR}(F))\backslash GL(H_1^{DR}(X)(\mathbb{A}_F))$.

π_1 reciprocity

$$k \in H^1(G_F, T_n) \mapsto \text{functions on } X(\mathbb{A}_F)$$

via functions on

$$H^1(G_F, U_n)\backslash {\prod}' H^1(G_v, U_n).$$