Local Fields assignments.

Problem 1 (for 22/10). Fix a prime number p, and write $|\cdot|$ for the p-adic absolute value on $\mathbb{Q}$, say with $\alpha = 1/p$. (So $|p^n \frac{a}{b}| = p^{-n}$.)

(1) Compute $|6!|$ for every prime p.

(2) Say that a sequence $(a_n)_{n=1}^{\infty}$ of rational numbers has a limit $a \in \mathbb{Q}$ if $|a_n - a| \to 0$ as $n \to \infty$. For $p = 2$, prove that the two sequences

$$9, 99, 999, 9999, \ldots \qquad \text{and} \qquad 5, 55, 555, 5555, \ldots$$

converge and find their limits.

(3) If $x \in \mathbb{Q}$ satisfies $|x| < 1$ (e.g. $x \in \mathbb{Z}$ is divisible by p), prove that

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x},$$

in the sense that the partial sums in the left-hand side tend to the right-hand side; if $|x| \geq 1$, prove that the series diverges. For example, when $p = 2$,

$$1 + 2 + 4 + 8 + 16 + 32 + \ldots = -1.$$

Problem 2 (for 29/10). Suppose $k = \bar{k}$ is an algebraically closed field, and let $K = k(t)$ be a field of rational functions in one variable.

(1) Prove that every normalised discrete valuation on K which is trivial on k (i.e. $v(a) = 0$ for $a \in k^*$) is either of the form v_a for some $a \in k$ ("order of vanishing at a") or is $v_\infty(p/q) = \deg q - \deg p$.

(2) What happens if k is not algebraically closed?

Problem 3 (for 5/11). Suppose p is an odd prime.

(1) Prove that for every $a \in \mathbb{Z}_p^\times$ the sequence $(a^{p^n})_{n\geq 1}$ is Cauchy, and hence converges. Denote its limit by $[a]$. Show that $[a] \equiv a \mod p$.

(2) Show that $[a] = 1$ when $a \equiv 1 \mod p$, and deduce that $a \mapsto [a]$ is an injective group homomorphism

$$\frac{\mathbb{Z}_p^\times}{1 + p\mathbb{Z}_p} \quad \cong \quad (\mathbb{Z}/p\mathbb{Z})^\times \quad \xrightarrow{[\cdot]} \quad \mathbb{Z}_p^\times.$$

The map $a \mapsto [a]$ can be viewed as a (unique) way to lift elements from the residue field $(\mathbb{Z}/p\mathbb{Z})^\times$ back to $\mathbb{Z}_p^\times$, in a multiplicative way. It is called the *Teichmüller lift*, and it shows that $\mathbb{Q}_p$ contains $(p-1)$th roots of unity.

Problem 4 (for 12/11). Prove that the equation

$$x^2 + y^2 = 3$$

(1) has solutions in $\mathbb{F}_p$ for every prime $p > 3$. (Hint: the sets $\{x^2 : x \in \mathbb{F}_p\}$ and $\{3 - x^2 : x \in \mathbb{F}_p\}$ cannot have empty intersection.)
(2) has solutions in $\mathbb{Q}_p$ for every prime $p > 3$ (and in $\mathbb{R}$ for that matter).
(3) has no solutions in $\mathbb{Q}_2$ or in $\mathbb{Q}_3$.

Problem 5 (for 19/11). For $n \geq 2$ write $\zeta_n = e^{2\pi i n}$, a primitive nth root of unity in $\mathbb{C}$. You may use that $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is a Galois extension of degree $\phi(n)$, the Euler phi function of n.

(1) Prove that $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is naturally isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$.
(2) For every prime p, show that $\mathrm{Gal}(\bigcup_{m\geq 1} \mathbb{Q}(\zeta_{p^m})/\mathbb{Q}) \cong \mathbb{Z}_p^\times$ as groups.
(3) Similarly, show that $\mathrm{Gal}(\bigcup_{n\geq 1} \mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \hat{\mathbb{Z}}^\times$.

Problem 6 (for 26/11). Prove that $\mathbb{R}, \mathbb{Q}_2, \mathbb{Q}_3, \mathbb{Q}_5, \ldots$ are pairwise non-isomorphic as fields (no topology!). [Hint: Problem 4 may give you a plan.]

Problem 7 (for 3/12). Let p be an odd prime, $K = \mathbb{Q}_p$ and $\eta \in \mathbb{Z}_p^\times$ a unit for which $\bar{\eta} \in \mathbb{F}_p^\times$ is a quadratic non-residue. Let

$$L = K(\text{roots of } x^4 - \eta\, p^2).$$

(1) Proof that $e_{L/K} = f_{L/K} = 2$.
(2) Determine $\mathrm{Gal}(L/K)$ and list all intermediate fields $K \subset M \subset L$. Note: $p \equiv 3 \mod 4$ and $p \equiv 1 \mod 4$ give two different answers.

Problem 8 (for 10/12). Denote by $\zeta = \zeta_8$ a primitive 8th root of unity, that is a root of $x^4 + 1$.

(1) Prove that $\mathbb{Q}_2(\zeta) = \mathbb{Q}_2(\sqrt{2}, \sqrt{-1})$.
(2) Find the minimal polynomial of $\pi = \zeta - 1$ over $\mathbb{Q}_2$, and deduce that $\mathbb{Q}_2(\zeta)/\mathbb{Q}_2$ is totally ramified, and π is a uniformiser.
(3) Determine the ramification groups $G_i \subset \mathrm{Gal}(\mathbb{Q}_2(\zeta)/\mathbb{Q}_2) \cong C_2 \times C_2$.