§ Global l-adic representations & compatible systems

Unifying theory of L-functions

Dirichlet char. $\searrow$
Artin rep. $\to$
Ell. curves / $\mathbb{Q}$ $\to$
H^i_{et} of $V/\mathbb{Q}$ $\nearrow$
$\uparrow$
modular forms

$V = (V_\ell)_{\ell \text{ prime}}$

compatible system (= "motive") $\longrightarrow$ $L(s)$

attempt to understand all interesting L-functions via some sort of cohomology theory

Fix $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$

$$G_{\mathbb{Q}_p} = \underset{\substack{\cup \\ I_p,\ \mathrm{Frob}_p}}{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \overset{\mathrm{Res}}{\hookrightarrow} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = G_{\mathbb{Q}}$$

Def A <u>(global) l-adic rep</u>. is a cont. hom.

$$\rho_\ell : G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{Q}_\ell) = GL(V_\ell)$$

A <u>compatible system</u> of l-adic reps $V = (V_\ell)_{\ell \text{ prime}}$:

approximation to having one $\mathbb{Q}$-v.sp. with $G_{\mathbb{Q}}$-action

(1) V_ℓ unramified at $p \notin S \cup \{\ell\}$, S fixed finite set of primes
$[\rho_\ell(I_p) = 1]$

(2) $\forall p$ <u>local polys</u>

$$P_p(T) = \det(1 - T \cdot \mathrm{Frob}_p^{-1} \mid V_\ell^{I_p}) \in \mathbb{Q}_\ell[T]$$

$\deg = n$ if $p \notin S \cup \{\ell\}$
$\leq n$ always

are in $\mathbb{Q}[t]$ and independent of $\ell \neq p$.

Let $L(V,s) = \prod_p P_p(p^{-s})^{-1}$ degree n L-fncn

<u>Serre</u> $\to$ $L(V,s)$ determines V up to semisimplification

<u>Rmk</u> May replace $G_{\mathbb{Q}}$ by G_F, $F/\mathbb{Q}$ finite "field of definition"

$(\mathbb{Q}_\ell)_\ell$ by $(K_\lambda)_\lambda$, $K/\mathbb{Q}$ finite "field of coefficients"
l-adic λ-adic

<u>Ex</u> (Artin reps) $\rho : G_{\mathbb{Q}} \twoheadrightarrow \underset{\text{finite}}{G} \longrightarrow GL_n(\mathcal{V})$ $\mathcal{V}$ fin.dim $\mathbb{Q}$- (or K-, $\mathbb{C}$-) vector space

$$V_\ell := \mathcal{V} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$$

$\leadsto$

$L(V,s)$ Artin L-fncn degree n, |roots| = 1

Ex (Dirichlet chars) $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to K^\times$ primitive

$$\rho_\chi : G_\mathbb{Q} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times \xrightarrow{\chi} K^\times = GL(V)$$

$$\mathrm{Frob}_p \leftarrow p$$

V unramified at $p \iff p \nmid m$

$$P_p(T) = \begin{cases} 1-\chi(p)p^{-s} & p\nmid m \\ 1 & p \mid m \end{cases}$$

$$L(V,s) = \prod_p \frac{1}{1-\chi(p)p^{-s}} = \sum_{n\geq 1} \frac{\chi(n)}{n^s} = L(\chi, s).$$

and all abelian reps of $G_\mathbb{Q}$ are direct sums of these

Ex (Cyclotomic character) $G_\mathbb{Q} \circlearrowleft$ ℓ-power roots of 1 $\xrightarrow{\varprojlim} V_\ell =: \mathbb{Q}_\ell(1)$

$$P_p(V,T) = 1 - pT$$

degree 1, $S = \emptyset$
$|\text{roots}| = \frac{1}{p}$

$$L(V,s) = \prod_p \frac{1}{1-p\cdot p^{-s}} = \zeta(s-1).$$

Ex (Elliptic curves)

$V = (V_\ell E^*)_\ell \rightsquigarrow L(E,s)$

degree 2
$|\text{roots}| = \frac{1}{\sqrt{p}}$ $(p \notin S)$

Ex (étale cohomology) $X/\mathbb{Q}$ non-sing. proj. var., $0 \leq i \leq 2\dim X$

$\rightsquigarrow H^i(X) = H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ $\mathbb{Q}_\ell$-v.sp., $\dim = b_i =$ i^{th} Betti number of $X(\mathbb{C})$

(1) $S = \{$primes of bad red. of $X\}$

(2) Known to be compatible at $p \notin S$;
for H^0, H^1, curves, ab.vars for $p \in S$ as well.

if compatible $\rightsquigarrow$ $L(H^i(X), s)$

degree b_i
$|\text{roots}| = p^{-i/2}$ $(p \notin S)$

Ex $H^0(X) = \mathbb{Q}_\ell[\text{connected components of } X/\overline{\mathbb{Q}}]$ $\rightsquigarrow$ Artin L-fnc.

$\circlearrowleft$
$G_\mathbb{Q}$

just $\zeta(s)$ if $X_{\overline{\mathbb{Q}}}$ connected

Ex $X/\mathbb{Q}$ curve

$$H^1(C) \cong H^1(\mathrm{Jac}\, C) \cong (V_\ell \, \mathrm{Jac}\, C)^*$$

degree $2g$
$|\text{roots}| = \frac{1}{\sqrt{p}}$

$(p \notin S)$
primes of bad reduction

Ex (Poincaré duality) $d = \dim X$

$$H^{d+i}(X) \cong H^{d-i}(X) \otimes \mathbb{Q}(i)$$

Main example

$X = C/\mathbb{Q}$ curve

$H^0(C) = \mathbb{Q}_\ell \rightsquigarrow L(H^0(C), s) = \zeta(s)$ deg 1, $|\text{roots}| = 1$

$H^1(C) = V_\ell \operatorname{Jac} C^* \rightsquigarrow L(H^1(C), s) =: L(C, s)$ deg 2g, $|\text{roots}| = \sqrt{\ }$

$H^2(C) = \mathbb{Q}_\ell(1) \rightsquigarrow L(H^2(C), s) = \zeta(s-1)$ deg 1, $|\text{roots}|$.

Rmk $A/\mathbb{Q}$ abelian variety (e.g. E, $\operatorname{Jac} C$) $\rightarrow$

$L(A, s)$ determines $V_\ell(A)$ determines A up to isogeny (Faltings)

Rmk Have standard constructions for all compatible systems:

$\oplus$, $\otimes$, Ind, Res

and Artin formalism for L-fncs $\left[\begin{array}{l} L(V \oplus V', s) = L(V, s) L(V', s) \\ L(\operatorname{Ind} V, s) = L(V, s) \end{array} \right]$.

Rmk Conjecturally, all $L(H^i(V), s)$ have

- meromorphic continuation to $\mathbb{C}$ (AC)
- fun. eq. $s \leftrightarrow i+1-s$ (FE)
- zeroes on $\operatorname{Re} s = \frac{i+1}{2}$ (RH)
- special value conjs at $s \in \mathbb{Z}$ (BSD & co.)

From now on, $X = C$ curve.

§ H^1 of curves over local fields

$K/\mathbb{Q}_p$ finite, $\mathcal{O}_K, \pi, k$

C/K curve, genus ≥ 1.

$C/K \overset{\text{WANT}}{\rightsquigarrow} H^1(C) = H^1_{et}(C_{\bar{K}}, \mathbb{Q}_\ell)$ $\dim = 2g$ ($\rightsquigarrow L(C,s)$ over $\mathbb{Q}$)

$\circlearrowleft G_K$