① $H^1(C)$ from P

② P from regular model

③ Computing regular models

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① Recall

$$P(C/K, T) = \det\left(1 - \mathrm{Frob}_K^{-1}\, T \mid H^1(C_{\bar K}, \mathbb{Q}_\ell)^{I_K}\right) \quad \in \mathbb{Z}[T], \text{ indep. of } \ell$$

<u>Thm 1</u> $P(C/K', T)$ for all K'/K finite determine $H^1(C)$ uniquely

<u>Pf</u> Rep. theory (cf. Exc VD #1)

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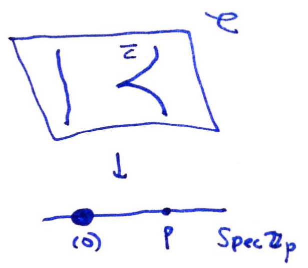
② <u>Def</u> A regular model of C/K is a regular 2-dim scheme $\mathcal{C}$, proper $/\mathcal{O}_K$, $\mathcal{C} \otimes_{\mathcal{O}_K} K \cong C$.

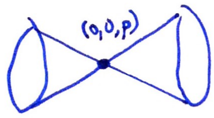
<u>Ex</u> $C/\mathbb{Q}_p$: $y^2 = x^3 + p$ (any p)

$\mathcal{C}/\mathbb{Z}_p$: $y^2 = x^3 + p$ regular model



<u>Ex</u> $C/\mathbb{Q}_p$: $y^2 = x^3 + p^2$ $(p \neq 2)$

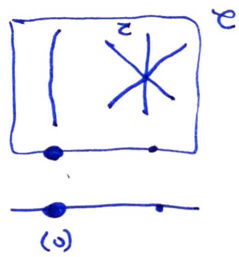
$/\mathbb{Z}_p$: $y^2 = x^3 + p^2$ not regular



Over $\mathbb{Q}_p$,

$$y^2 = x^3 + p^2 \quad \cong \quad 4xy(x+y) = p \quad \leftarrow \text{regular model } \mathcal{C}/\mathbb{Z}_p)$$

(Type IV reduction)



<u>Thm 2</u> $C(K) \neq \emptyset$, $\mathcal{C}/\mathcal{O}_K$ regular model, special fibre $\bar{\mathcal{C}}/k$. Then

1) $H^1(C)^{I_K} \cong H^1(\bar{\mathcal{C}})$ as G_k-modules.

2) $Z(\bar{\mathcal{C}}/k, T) = \dfrac{P_1(T)}{P_0(T)\, P_2(T)}$; $P_1(T) = P(C/K, T)$ by 1)

want

can compute by point counting

$H^0(\bar{\mathcal{C}}) = \mathbb{Q}_\ell$ [connected comps of $\bar{\mathcal{C}}_{\bar k}$] $= \mathbb{Q}_\ell$ $\rightsquigarrow$ $P_0(T) = 1 - T$

$H^2(\bar{\mathcal{C}}) = \mathbb{Q}_\ell(1)$ [irr. comps of $\bar{\mathcal{C}}_{\bar k}$] $\rightsquigarrow$ $P_2(T)$ [$= 1 - qT$ if $\bar{\mathcal{C}}_{\bar k}$ irreducible]

(3) Computing regular models [example]

$$C : \underbrace{\sum a_{ij} x^i y^j}_{f} = 0$$

Δ = convex hull of $\{(i,j) \mid a_{ij} \neq 0\} \subseteq \mathbb{R}^2$

$\tilde{\Delta}$ = lower convex hull of $\{(i,j,v(a_{ij}))\} \subseteq \mathbb{R}^3$

$\mathrm{proj}(\tilde{\Delta})$ breaks Δ into 2-faces F_i, 1-faces L_i

denominators

δ_F = common denom of v on $F \cap \mathbb{Z}^2$

δ_L — " — $L \cap \mathbb{Z}^2$

reductions

$f_F[x,y]$, $f_L \in k[t]$

slopes

$L = F_1 \cap F_2$ (inner) or $L \subseteq F_1$ (outer)

pick $P \in \mathbb{Z}^2$ right off L in the F_1-direction

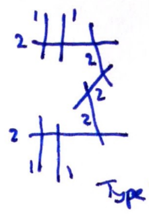
$s_L^1 = \delta_L v_{F_1}(P)$

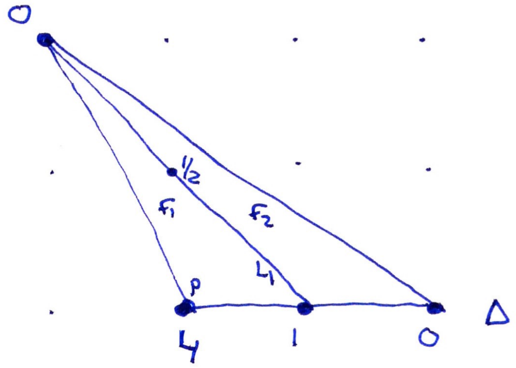
└ linear ext. of $v|_{F_1}$ to $\mathbb{R}^2$

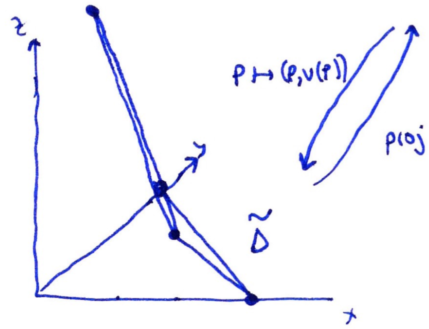
$$s_L^2 = \begin{cases} \delta_L v_{F_2}(P) & \text{inner} \\ \lfloor s_L^1 \rfloor - 1 & \text{outer} \end{cases}$$

Ex $y^2 - (x^3 - px^2 + p^4x) = 0$ $/\mathbb{Q}_p$

⇝ regular model







$f_{F_1} = y - x - x^2$

$f_L = t - 1$ (all f_{L_i} linear)

$L_1 = F_1 \cap F_2$, $P = (1,0)$

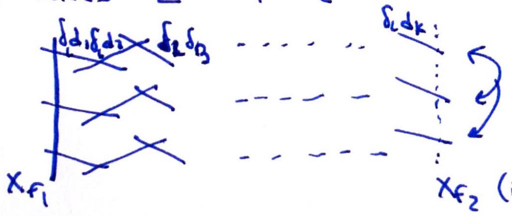
⇝ $s_L^1 = 2 \cdot 4$

$s_L^2 = 2 \cdot 2$

Thm 3 If all $f_F = 0$ in $\mathbb{G}_m^2$ (└ $\mathbb{A}^2 \setminus \{x,y\text{-axis}\}$), all $f_L = 0$ in $\mathbb{G}_m$ (└ $\mathbb{A}^1 \setminus \{0\}$) are smooth, then

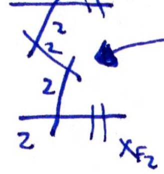
C/K has a regular model (with normal crossings) $\mathcal{C}/\mathcal{O}_K$ with special fibre:

- 2-faces F ⇝ components $X_F/k : f_F = 0$ of mult. δ_F and genus $\#\{P \in \mathrm{interior}(F) \cap \mathbb{Z}^2 \mid v(P) \in \mathbb{Z}\}$
- 1-faces $L = F_1 \cap F_2$ (or outer $\subseteq F_1$) ⇝ chains of $\mathbb{P}^1$'s $/\bar{k}$



chains $\xleftrightarrow{1:1}$ roots of f_L in $\bar{k}^*$, as a G_k-set.

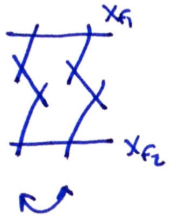
$$s_L^1 = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \cdots > \frac{n_k}{d_k} > \frac{n_{k+1}}{d_{k+1}} = s_L^2 \qquad \begin{vmatrix} n_i & n_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = 1$$

In example above,  $\leftarrow$ $s_{F_1} = \frac{8}{1} > \frac{7}{①} > \frac{6}{①} > \frac{5}{①} > \frac{4}{1} = s_{F_2}$

$\times\, \delta_\ell = 2$

other 4 1-faces give 

Exc Over $\mathbb{Q}_p(\sqrt{p})$, $p \neq 2$



(Type I_8)

Swapped by G_k $\iff$ $p \equiv 3 \bmod 4$ ($\sqrt{-1} \notin \mathbb{F}_p$)

(potentially mult. red. becoming split or nonsplit mult.)

Cor Under the assumptions of Thm 3, decompose (uniquely)

$$H^1_{et}(C_{\bar{K}}, \mathbb{Q}_\ell) \cong H^{tame}_{ab} \oplus (H^{tame}_{toric} \otimes Sp_2) \oplus H^{wild}$$

tame, finite I_K-image — trivial I_{tame}-invariants

Write $\Delta(\mathbb{Z})$ = interior of $\Delta \cap \mathbb{Z}^2$ ($|\Delta(\mathbb{Z})| = g(C)$ by Baker's Thm)

$\Delta^{tame}_{toric} = \{P \in \Delta(\mathbb{Z}) \mid p \nmid \operatorname{den} v(P),\ P \in \text{some 1-face}\}$

$\Delta^{tame}_{ab} = \{P \in \Delta(\mathbb{Z}) \mid p \nmid \operatorname{den} v(P),\ P \notin \text{any 1-face}\}$

$\Delta^{wild} = \{P \in \Delta(\mathbb{Z}) \mid p \mid \operatorname{den} v(P)\}$

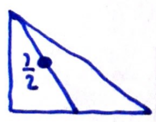
Every $P \in \Delta^{tame}_{*}$ with $v(P) = \frac{n}{d}$ determines a character $\chi_P = \chi_d^n : I_{tame} \to \bar{\mathbb{Q}}_\ell^\times$ (χ_d some fixed order d character).

Then

$$H^{tame}_{ab} \cong \bigoplus_{P \in \Delta^{tame}_{ab}} (\chi_P \oplus \chi_{P^{-1}})$$

$$H^{tame}_{toric} \cong \bigoplus_{P \in \Delta^{tame}_{toric}} \chi_P \otimes Sp_2 \qquad \text{as } I_K\text{-modules.}$$

$$\dim H^{wild} = 2\,|\Delta^{wild}|.$$

Ex 

$p \neq 2 \Rightarrow H^1 \cong \chi_2 \otimes Sp_2$ (pot. mult. red.)

ramified quadratic

$p = 2 \Rightarrow H^1$ wildly ramified. (could be pot. good or pot. mult!)