Generic Rank of a Family of Elliptic Curves

Francesca Bianchi, Matthew Bisatt, Julie Desjardins, Daniel Kohen, Carlo Pagano, Soohyun Park

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Motivation

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A family of elliptic curves $\mathcal{E}$ is given by the equation

$$\mathcal{E} : y^2 + a_1(T)xy + a_3(T)y = x^3 + a_2(T)x^2 + a_4(T)x + a_6(T)$$

with $a_i(T) \in \mathbb{Z}[T]$. 

We denote by $E_t$ the curve given by this equation when $T$ is replaced by $t \in \mathbb{Q}$. We call $E_t$ the fibre at $t$ and is an elliptic curve except for finitely many exceptions.

We assume $a_1 = a_3 = 0$ with $\deg a_i \leq 2$ for $i$ even.
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**Definition**

The **generic rank** of \( \mathcal{E} \) (denoted by \( \text{rk}(\mathcal{E}(\mathbb{Q}(T))) \)) is the rank of \( \mathcal{E} \) as an elliptic curve over \( \mathbb{Q}(T) \).
**Definition**

The generic rank of $\mathcal{E}$ (denoted by $\text{rk}(\mathcal{E}(\mathbb{Q}(T))))$ is the rank of $\mathcal{E}$ as an elliptic curve over $\mathbb{Q}(T)$.

**Theorem (Silverman)**

We have $\text{rk}(\mathcal{E}_t) \geq \text{rk}(\mathcal{E}(\mathbb{Q}(T)))$ for all but finitely many $t \in \mathbb{Q}$. 

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Julie's group

Generic Rank

Trieste 2017
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  Q1: What if $k$ is not algebraically closed (e.g. $k = \mathbb{Q}$)?

- Q2: Is there an elliptic surface $\mathcal{E}$ over $\mathbb{Q}$ with generic rank 0 such that every fibre $\mathcal{E}_t$ has positive rank? (Cassels)
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- When $K$ is a number field, it is known that

  $$\#\{t \in \mathbb{P}^1 \mid \text{rg}(\mathcal{E}_t) \neq 0\} = \infty \iff \mathcal{E}(K) \text{ is dense}$$
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  Q3: Suppose there is no elliptic curve $E$ over $\mathbb{Q}$ such that $\mathcal{E} \cong E \times \mathbb{P}^1$.

  Is $\mathcal{E}(\mathbb{Q})$ Zariski dense?
**Computing the generic rank**

**Conjecture (Nagao)**

The rank of $\mathcal{E}$ over $\mathbb{Q}(T)$ is

$$r_{\mathcal{E}} = \lim_{X \to \infty} \frac{1}{X} \sum_{p \leq X} -A_{\mathcal{E}}(p) \log p,$$

where $p$ runs through all primes $p \leq X$ and

$$A_{\mathcal{E}}(p) := \frac{1}{p} \sum_{t=0}^{p-1} a_{\mathcal{E}_t}(p),$$

where $a_{\mathcal{E}_t}(p) = p + 1 - \#\mathcal{E}_t(\mathbb{F}_p)$. 
Previous work

We follow methods previously used by Bettin, David, and Delaunay.

\[
y^2 = x^3 + a_2(x^2) + a_4(x) + a_6
\]

\(a_i \in \mathbb{Z}[T]\), with no multiplicative reduction except possibly at infinity.
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Problems we considered

Elliptic surfaces with multiplicative reduction at finite places

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- Density of rational points
More on Nagao’s conjecture

Nagao’s Conjecture

Assume $\mathcal{E}$ is not constant. Then the generic rank is

$$\lim_{X \to \infty} \frac{1}{X} \sum_{p \leq X} \frac{-\log p}{p} \sum_{t=0}^{p-1} a_{\mathcal{E}(t)}(p),$$

where $a_{\mathcal{E}(t)}(p)$ is the trace of Frobenius at $p$ of the specialisation at $t$. 

This is true in the case of rational elliptic surfaces, due to Rosen and Silverman.
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Proposition (BBDKPP)

Let $k \in \mathbb{Q}^\times$ and consider the family of elliptic surfaces

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From Nagao’s conjecture, we find

$$\text{rank } \mathcal{E}(\mathbb{Q}(T)) \leq 1,$$

with equality if and only if $k \in \pm (\mathbb{Q}^\times)^2$. 

An Example of What we Found
Proposition (BBDKPP)

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From Nagao’s conjecture, we find

$$\text{rank} \mathcal{E}(\mathbb{Q}(T)) \leq 1,$$

with equality if and only if $k \in \pm (\mathbb{Q}^\times)^2$. Moreover, the generating section is

$$(0, \sqrt{kT}) \quad \text{if } k \text{ is a square};$$

$$( -k, \sqrt{(-k)^3} ) \quad \text{if } -k \text{ is a square}.$$
**Shioda-Tate Formula**

\[
\text{rank } \mathcal{E}(\overline{\mathbb{Q}(T)}) = \text{rank } NS(\mathcal{E}) - 2 - \sum_v (m_v - 1)
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**Generic rank over \( \overline{\mathbb{Q}}(T) \)**

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**In our case**

Let \( \mathcal{E}_k : y^2 = x^3 + T^2x + kT^2 \). Then \( \Delta(\mathcal{E}_k) = -16T^4(4T^2 + 27k^2) \),

\[
j(\mathcal{E}_k) = 1728 \frac{4T^2}{4T^2 + 27k^2}.
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- At \( T \), we have type IV \((m_v = 3)\);
- At the linear factors of \((4T^2 + 27k^2)\), we have type \( I_1 \) \((m_v = 1)\);
- At \( \infty \), we have type \( I_0^* \) \((m_v = 5)\).
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So \( \text{rank } \mathcal{E}_k(\overline{\mathbb{Q}}(T)) = 10 - 2 - (3 - 1) - 2(1 - 1) - (5 - 1) = 2 \).
Theorem (BBDKPP)

Consider the non-isotrivial elliptic surface

\[ \mathcal{E} : y^2 = x^3 + a_4(T)x + a_6(T), \]

with \( \deg a_i \leq 2 \) such that there are exactly two fibres of multiplicative reduction over \( \overline{\mathbb{Q}} \).
Classification when $a_2 = 0$

**Theorem (BBDKPP)**

Consider the non-isotrivial elliptic surface

$$\mathcal{E} : y^2 = x^3 + a_4(T)x + a_6(T),$$

with $\deg a_i \leq 2$ such that there are exactly two fibres of multiplicative reduction over $\overline{\mathbb{Q}}$. Then $\mathcal{E}$ belongs to one of the following families:

- $y^2 = x^3 + kx + T$ with $k \in \mathbb{Q}^\times$;
- $y^2 = x^3 + (aT + b)x + (aT^2 + bT)$ where $a \neq 0$ and $b \neq a^2/27$;
- $y^2 = x^3 + P(T)x + kP(T)$ for some quadratic polynomial $P$ and $k \in \mathbb{Q}^\times$ such that $4P(T) + 27k^2$ is nonsquare in $\overline{\mathbb{Q}}[T]$. 
The isotrivial elliptic surface

\[ \mathcal{E} : y^2 = x^3 + T \]

has rank(\(\mathcal{E}(\mathbb{Q}(T))\)) = 0.

However, it has infinite subfamilies of positive rank. In particular, the subfamily of elliptic curves (given by Nagao)

\[ \mathcal{E}_s : y^2 = x^3 + (s^2 - m^3) \]

has generic rank 1 for any fixed \(m \in \mathbb{Z} \setminus 0\).
Isotrivial elliptic surface

**Example**

The isotrivial elliptic surface

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Conjecture (BBDKPP)\[\text{The elliptic surface}\]
\[E: y^2 = x^3 + 15(27T^6 + 1)\]
has positive generic rank.

Family of constant root number \(W(E_t) = -1\) for all \(t \in \mathbb{Q}\) found by Julie.

Our method doesn't work since \(\text{deg } a_i\) too large. :-(
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Open questions and possible future work

- Use known families with constant root number to guess interesting subfamilies of elliptic curves with high rank?
- Generic rank when $\deg a_i$ is high
Thank you!


