Hypergeometric motives of low degrees

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Workshop on Arithmetic of Hyperelliptic Curves

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Equation of Gauss

Hypergeometric series

Let $a, b, c \in \mathbb{C}$. Consider a differential equation

$$t(t-1)y'' + ((a+b+1)t - c)y' + aby = 0$$

Equivalent form: $\mathcal{D} = t(\theta + a)(\theta + b) - \theta(\theta + c - 1)$, where $\theta = t\frac{d}{dt}$

$$\mathcal{D}y = 0$$

Parameters a, b, c are called hypergeometric and they form a pair of tuples (a, b) and $(1, c)$.

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```
F<a,b,c>:=FunctionField(Rationals(),3);
F0<t>:=RationalDifferentialField(F);
RD<D>:=DifferentialOperatorRing(F0);
RH<H>,mp:=ChangeDerivation(RD,t);
op:=t*(H+a)*(H+b)-H*(H+c-1);
1/t*Inverse(mp)(op);
```

Equation of Gauss

Hypergeometric series

For c not integral we can write down the following basis of solutions:

$$t(t-1)y'' + ((a+b+1)t - c)y' + aby = 0$$

Define

$$F\left(\begin{array}{cc} a & b \\ & c \end{array} \mid t\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} t^n$$

where

$$(x)_n = x \cdot (x+1) \cdot \ldots \cdot (x+n-1)$$

A basis of (two independent) solutions to this differential equation around $t = 0$ is

$$y = F\left(\begin{array}{cc} a & b \\ & c \end{array} \mid t\right)$$

$$y = t^{1-c} F\left(\begin{array}{cc} a+1-c & b+1-c \\ & 2-c \end{array} \mid t\right)$$

Hypergeometric equations

For positive integer d we consider two tuples $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$ and $\beta_1, \ldots, \beta_d \in \mathbb{C}$. We define a hypergeometric operator

$$\mathcal{D}(\alpha, \beta) := t \cdot (\theta + \alpha_1) \cdot \ldots \cdot (\theta + \alpha_d) - (\theta + \beta_1 - 1) \cdot \ldots \cdot (\theta + \beta_d - 1)$$

Differential equation $\mathcal{D}(\alpha, \beta)y = 0$ has locally d independent solutions

Around 0 one can describe the basis in terms of hypergeometric functions (β_i distinct modulo 1)

$${}_dF_{d-1}\left(\begin{array}{ccc} \alpha_1 & \ldots & \alpha_d \\ \beta_1 & \ldots & \beta_{d-1} \end{array} \mid t\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdot \ldots \cdot (\alpha_d)_n}{(\beta_1)_n \cdot \ldots \cdot (\beta_{d-1})_n n!} t^n$$

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Basis of solutions: for $1 \leq i \leq d$

$$t^{1-\beta_i} {}_dF_{d-1}\left(\begin{matrix} \alpha_1 - \beta_i + 1 & \ldots & \alpha_d - \beta_i + 1 \\ \beta_1 - \beta_i + 1 & \ldots\check{}\ldots & \beta_{d-1} - \beta_i + 1 \end{matrix} \mid t \right)$$

Monodromy groups

Hypergeometric equation $D(\alpha, \beta)$ come with ***differential Galois group*** (algebraic group) and ***monodromy group*** (discrete subgroup).

Differential Galois groups $DG(\alpha, \beta)$ of $D(\alpha, \beta)$ were described and classified by Beukers and Heckman.

$DG(\alpha, \beta)$ of $D(\alpha, \beta)$ is finite iff the elements of α, β interlace.

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The monodromy group can be computed by a theorem of Levelt: for $\alpha_i - \beta_j \notin \mathbb{Z}$ (hence the system is irreducible)

$$p_\alpha = \prod (t - exp(2\pi i \alpha_k)), \quad p_\beta = \prod (t - exp(2\pi i \beta_j))$$

Let A be the companion matrix of p_α and B of p_β. Then $h_\infty = A$, $h_0 = B^{-1}$ and $h_1 = A^{-1}B$.

Monodromy group $M(\alpha, \beta)$ is spanned by h_0, h_∞. Zariski closure of $M(\alpha, \beta)$ gives $DG(\alpha, \beta)$.

Transition to finite fields

$$F(\alpha, \beta|z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_d)_n} \ z^n.$$

$$\frac{\Gamma(\beta_1) \cdots \Gamma(\beta_d)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_d)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + n) \cdots \Gamma(\alpha_d + n)}{\Gamma(\beta_1 + n) \cdots \Gamma(\beta_d + n)} \ z^n \ .$$

$$\sum_{n=0}^{\infty} \prod_{i=1}^{d} \left(\frac{\Gamma(\alpha_i + n)(1 - \beta_i - n)}{\Gamma(\alpha_i)\Gamma(1 - \beta_i)} \right) \ (-1)^{dn} z^n \ .$$

Transition to finite hypergeometric sums

Let χ be any *multiplicative* character of finite field $\mathbb{F}_q^\times$ with values in $\mathbb{C}^\times$. We fix an *additive* character ψ. A **Gauss sum** $g(\chi, \psi)$ is

$$g(\chi, \psi) = \sum_{x \in \mathbb{F}_q^\times} \chi(x)\psi(x).$$

For a fixed generator ω of the group of characters we denote by $g(m)$ the sum $g(\omega^m, \psi)$.

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Definition (Finite hypergeometric sum, BCM)

Let α, β in $\mathbb{Q}^d$ such that $\alpha_i \not\equiv \beta_j (\mathrm{mod}\ \mathbb{Z})$ and $\mathrm{q}\alpha_i$ and $\mathrm{q}\beta_j$ are integral. We define for any $t \in \mathbb{F}_q$, $\mathrm{q} = q - 1$

$$H_q(\alpha, \beta | t) = \frac{1}{1-q} \sum_{m=0}^{q-2} \prod_{i=1}^{d} \left(\frac{g(m + \alpha_i \mathrm{q}) g(-m - \beta_i \mathrm{q}))}{g(\alpha_i \mathrm{q}) g(-\beta_i \mathrm{q})} \right) \omega((-1)^d t)^m .$$

These sums with different normalisation were considered by Katz, Greene and McCarthy.

Finite geometric sums a la Greene, Katz and Beukers-Cohen-Mellit

We say that the sum $H_q(\alpha, \beta)$ is *defined over* $\mathbb{Q}$ if polynomials $A(x) = \prod_{j=1}^{d}(x - e^{2\pi i \alpha_j})$ and $B(x) = \prod_{j=1}^{d}(x - e^{2\pi i \beta_j})$ are defined over $\mathbb{Q}$ (actually $\mathbb{Z}$). Then there exist integers $p_1, \dots, p_r$ and $q_1, \dots, q_s$ such that

$$\prod_{j=1}^{d} \frac{x - e^{2\pi i \alpha_j}}{x - e^{2\pi i \beta_j}} = \frac{\prod_{j=1}^{r} x^{p_j} - 1}{\prod_{j=1}^{s} x^{q_j} - 1} \, .$$

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$$\prod_{j=1}^{d} \frac{x - e^{2\pi i \alpha_j}}{x - e^{2\pi i \beta_j}} = \frac{\prod_{j=1}^{r} x^{p_j} - 1}{\prod_{j=1}^{s} x^{q_j} - 1} .$$

Theorem (Beukers-Cohen-Mellit, 2016)

$$H_q(\alpha, \beta | t) = \frac{(-1)^{r+s}}{1-q} \sum_{m=0}^{q-2} q^{-s(0)+s(m)} g(pm, -qm) \omega(\epsilon M^{-1} t)^m$$

where $g(pm, -qm) = g(p_1 m) \cdots g(p_r m) g(-q_1 m) \cdots g(-q_s m)$, $M = \prod_{j=1}^{r} p_j^{p_j} \prod_{j=1}^{s} q_j^{-q_j}$ *and* $\epsilon = (-1)^{\sum_i q_i}$ *and* $s(m)$ *is the multiplicity of the zero* $e^{2\pi i m/\mathfrak{q}}$ *in* $GCD(A(x), B(x))$.

Canonical variety

Variety V_t attached to hypergeometric datum $(p_1, \ldots, p_r)$, $(q_1, \ldots, q_s)$

$$V_t : x_1 + x_2 + \cdots + x_r - (y_1 + \cdots + y_s) = 0, \quad t x_1^{p_1} \cdots x_r^{p_r} = y_1^{q_1} \cdots y_s^{q_s}$$

$$V_t \subset \mathbb{P}^{r+s-1}$$

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$$V_t \subset \mathbb{P}^{r+s-1}$$

Lemma (Beukers-Cohen-Mellit, 2016)

Assume that $\gcd(p_1, \ldots, p_r, q_1, \ldots, q_s) = 1$. *Let* $V_t(\mathbb{F}_q^\times)$ *be the set of points on* V_t *with coordinates in* $\mathbb{F}_q^\times$. *Then*

$$|V_t(\mathbb{F}_q^\times)| = \frac{1}{q}(q-1)^{r+s-2} + \frac{1}{q(q-1)} \sum_{m=0}^{q-2} g(pm, -qm)\omega(\epsilon t)^m,$$

Theorem (Beukers-Cohen-Mellit, 2016)

There exists a smooth compactification $\overline{V_t}$ of V_t such that

$$|\overline{V_t}(\mathbb{F}_q)| = P_{rs}(q) + (-1)^{r+s-1} q^{\min(r-1,s-1)} H_q(\alpha, \beta | Mt).$$

where P_{rs} is a polynomial (explicit).

This compactification might not be a minimal one. Subscheme $\overline{V_t} \setminus V_t$ is produced combinatorially but is quite difficult to work with.

For canonical varieties of dimension 2 we can obtain often a better compactification (minimal).

L-function datum

- Hypergeometric data (α, β) comes with **degree** and **weight**.
- degree = max (length(α), length(β))
- Fedorov proved that the connection of rank d on trivial holomorphic bundle over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ has a real polarizable variation of Hodge structures and gave a recipe for the **Hodge vector**
  (with $\alpha_m + \alpha_{d+1-m} \in \mathbb{Z}$, $\beta_m + \beta_{d-1} + 1 - m \in \mathbb{Z}$):

  $$\rho(j) := \#\{i : \alpha_i < \beta_j\} - j$$

  weight $= p_+ - p_-$, $p_+ = \max \rho(k)$, $p_- = \min \rho(k)$ and

  $$\operatorname{rk} H^{k-p_-,-k+p_+} = \#\rho^{-1}(k)$$

L-function datum

- One can compute the good factors of the L-function of **hypergeometric motive** $H(\alpha, \beta|t)$ (defined over $\mathbb{Q}$ for $t \in \mathbb{Q}$) using the hypergeometric formula $H_q(\alpha, \beta|t)$
- Bad factors correspond to primes p that divide α_i or β_j or numerator or denominator of $(t-1)/t$.
- Computation of L–function of the hypergeometric motive $H(\alpha, \beta|t)$ can be partially done now in MAGMA (Mark Watkins package).

Questions

- What can one say in general about those L-functions? (Rodriguez Villegas, Roberts, Watkins; Cohen, Kedlaya, Voight, Yui, . . .)
- Equivalently one can talk about a *motive* $X(\alpha, \beta|t)$ attached to this data. In what sense the motive is defined, e.g. is there an effective Chow motive. Can one compute the motivic Galois group when this motive varies in family?

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- Hypergeometric motives of weight 0 correspond to Artin representations.
  Hypergeometric motives of weight 1 originate from curves.
  Hypergeometric motives of weight 2 can be found in surfaces.
  ....

Motives of surfaces

X - smooth projective surface over a number field K

$H^2(X) = H^2_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ has pure weight 2 and the hypergeometric motive of weight 2 can be found essentially in the transcendental part (not algebraic) of this subspace.

From the theory of motives of surfaces this produces a Chow motive via a decomposition of the diagonal correspondence $[\Delta_X] = \sum_{0 \le i \le 4} \pi_i$ (Kahn-Murre-Pedrini):

$$h(X) = \sum_i h_i(X), \quad h_i(X) = (X, \pi_i, 0)$$

π_2 splits as $\pi_2^{alg} + \pi_2^{tr}$ and we have the decomposition

$$h_2(X) \cong h_2^{alg}(X) \oplus t_2(X)$$

where $h_2^{alg}(X) \cong h(\underline{\mathrm{NS}_X})(1)$ is the Artin motive associated to $\underline{\mathrm{NS}_X} = \mathrm{NS}(X \otimes_K K^{sep})_{\mathbb{Q}}$ ($\mathbb{Q}$-linear geometric Néron-Severi group with G_K-module structure).

Fibred surfaces

We consider a smooth projective irreducible surface X with relatively minimal fibration $X \to C$ over a field $k = \overline{k}$.

One has the intersection pairing on $NS(X)/tors$.
Generic fibre of genus 1 with a marked point provides a structure of elliptic surface.
For genus $g > 1$ we pass to the Jacobian of the generic fibre.
Genus 0 fibrations are helpful for the unirational implies rational argument.

There is a Shioda-Tate formula for the rank of the NS group

$$\operatorname{rk} \mathrm{NS}(X) = 2 + \sum_{v \in R} (m_v - 1) + rank(J(k(C))).$$

Singular fibres were classified by Kodaira in genus 1 case and in general it is a hard but computable task.

Motives coming from Artin representations

The monodromy group of the hypergeometric equation is finite, hence a differential Galois group is finite. This implies that hypergeometric series lies in certain finite algebraic extension of $\mathbb{Q}(t)$.

- If the motive comes from a variety V_t of dimension 0 we can explicitely see the Galois action on the closed points
- If V_t is a positive dimension vartiety then the motive is hidden in the subgroup of algebraic cycles in the middle etale cohomology of a suitable compactification.
- If V_t is a surface then we build a minimal regular model S_t and analyse the image of $NS(S_t)$ in $H^2_{et}(S_t, \mathbb{Q}_\ell)$.

Theorem (BN, 2017)

Let $H(\alpha, \beta|t)$ be a hypergeometric motive of degree d, $2 \leq d \leq 8$ and weight 0. Suppose that the canonical variety V_t of H is a surface.

Then there exists an elliptic (or hyperelliptic) relatively minimal fibration $S_t \to \mathbb{P}^1$ such that $H(\alpha, \beta|t)$ is an explicit Chow submotive of $NS(S_t)$.

$$|S_t(\mathbb{F}_q)| = \underbrace{1}_{H^0} + \underbrace{0}_{H^1} + \underbrace{f(q)}_{H^2 \ not\ HGM} + \underbrace{qH(\alpha, \beta|M_H t)}_{H^2\ HGM} + \underbrace{0}_{H^3} + \underbrace{q^2}_{H^4}$$

Case $[-1, 2, -3, -4, 6]$ of degree 3, weight 0.

Variety $V_t : tx_2^2 - x_1 x_3^3(-1 - x_1 - x_2 - x_3)^4 = 0$. We find an elliptic fibration

$$E_t : y^2 = x^3 - tx^2 + \frac{t^2(u-1)^2 u^4}{4}$$

over $\mathbb{Q}(t)(u)$. Reducible singular fibres at $u = 0$ (I_4) and $u = 1$ (I_2 non-split, with Galois action above $\mathbb{Q}(\sqrt{-t})$. Elliptic fibration $\mathcal{E}_t \to \mathbb{P}^1$ is rational and according to classification theorem of Shioda-Inose we have (generically) for $K = \overline{\mathbb{Q}(t)}$

$$E_t(K(u)) \cong A_1^* \oplus A_3^*$$

We use the fact that the Mordell-Weil group is spanned by points of the form $P = (au^2 + bu + c, \ldots)$. We can also use the map to singular fibres to restrict the coefficients a, b, c. Finally we solve a Groebner basis problem.

The following points span the Mordell-Weil group:

$$R_1 = (0, 1/2t(u-1)^2u^2)$$

and

$$Q_i = (a_i tu(u-1), \frac{a_i t\sqrt{-t}}{2}u(u-1)(u+\frac{2}{a}))$$

for $1 \leq i \leq 4$ such that a_i is a root of $a^4t + 4a^3t + 1$.

Degeneration: For $t \neq \frac{1}{27}$ we have $E_t(K(u)) \cong A_1^* \oplus A_3^*$ and otherwise $(A_1^*)^2 \oplus \langle \frac{1}{4} \rangle$.

Group $E_t(K(u))$ has index four sublattice spanned by R_1 and three points

$$P_i = (b_i u^2, (\frac{\sqrt{b_i t}}{2}u - \frac{t^2}{2\sqrt{b_i t}})u^2), \quad i = 1, 2, 3$$

where $4b^3 - bt + t^2 = 4\prod_{i=1}^{3}(b - b_i)$.

Theorem (BN, 2017)

Let $t \in \mathbb{Q}$ be general. The Galois module $H = H^2_{et}(\mathcal{E}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ isomorphic to

$$1^5 \oplus \rho_2 \oplus \rho_3$$

where ρ_2 is a two-dimensional representation attached to the quadratic character of $\mathbb{Q}(\sqrt{-t})$ and ρ_3 is the Artin representation associated with the space $\langle P_1, P_2, P_3 \rangle \otimes \mathbb{Q}_\ell$.

$$|\mathcal{E}_t(\mathbb{F}_q)| = 1 + (6 + (\frac{-t}{q}))q + \mathrm{Tr}\,\mathrm{Frob}_q \mid \rho_3 + q^2.$$

$$\mathrm{Tr}\,\mathrm{Frob}_q \mid \rho_3 = qH_q(\alpha, \beta | 27t).$$

$$P_i = (b_i u^2, (\frac{\sqrt{b_i t}}{2} u - \frac{t^2}{2\sqrt{b_i t}}) u^2), \quad i = 1, 2, 3$$

where $4b^3 - bt + t^2 = 4\prod_{i=1}^{3}(b - b_i)$.
We have that

$$\mathcal{D}(1/3, 2/3; 3/2) b = 0$$

Hypergeometric differential equation $\mathcal{D}(1/3, 2/3; 3/2) y = 0$ has two independent solutions around 0 with basis generated by

$$F_1 = {}_2F_1(\frac{1}{3}, \frac{2}{3}; \frac{3}{2} \mid z) \quad F_2 = z^{-1/2}{}_2F_1(-\frac{1}{6}, \frac{1}{6}; \frac{1}{2} \mid z)$$

so the roots x_1, x_2, x_3 of $x^3 - \frac{27}{4z} x + \frac{27}{4z}$ are

$$x_1 = F_1 \quad x_2 = -\frac{1}{2} F_1 + \frac{3\sqrt{3}}{2} F_2 \quad x_3 = -\frac{1}{2} F_2 - \frac{3\sqrt{3}}{2} F_2$$

So $x(P_i) = t\alpha_i(27t)$.

Differential equation $\mathcal{D}(\frac{1}{6}, \frac{3}{6}, \frac{5}{6}; \frac{3}{4}, \frac{5}{4})y = 0$ has three independent solutions around 0

$$G_1 = {}_3F_2(\frac{1}{6}, \frac{3}{6}, \frac{5}{6}; \frac{3}{4}, \frac{5}{4} \mid z)$$

$$G_2 = z^{\frac{1}{4}}{}_3F_2(\frac{5}{12}, \frac{9}{12}, \frac{13}{12}; \frac{5}{4}, \frac{6}{4} \mid z) \quad G_3 = z^{-\frac{1}{4}}{}_3F_2(-\frac{1}{12}, \frac{3}{12}, \frac{7}{12}; \frac{3}{4}, \frac{2}{4} \mid z)$$

The roots $\pm y_1, \pm y_2, \pm y_3$ of the polynomial $\frac{4}{27}zx^6 - x^2 + 1$ are

$$y_1 = G_1 \quad y_2 = \frac{1}{2\sqrt{2}3^{3/4}}(G_2 - 6\sqrt{3}G_3) \quad y_3 = \frac{\sqrt{-1}}{2\sqrt{2}3^{3/4}}(G_2 + 6\sqrt{3}G_3)$$

$$P_i = (tx_i(27t)u^2, u^2(\frac{ty_i(27t)}{2}u - \frac{t^2}{2ty_i(27t)})).$$

Hyperelliptic case

We analyse example in degree 6, weight 0: [-2,5,-7,-10,14]. Variety V_t with fibration determined by $u = \frac{x_3 x_4}{x_5^2}$ determines a smooth projective surface S_t with fibration $\pi : S_t \to \mathbb{P}^1$ and generic fibre

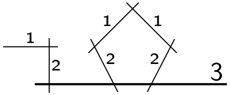
$$C_t : y^2 = 4x^5 + 4x^2 t u^2 - 4t^2 u + t^2$$

With choice of parameter $u' = \frac{x_1 x_3}{x_5^2}$ we can show that the surface S_t is unirational, hence by a theorem of Castelnuovo it is rational. For such genus g fibrations Saito proved that Picard rank satisfies

$$\rho(S_t) \leq 4g + 6$$

Shioda proved that the Jacobian $J = J(C_t)$ over $\mathbb{Q}(t)(u)$ satisfies $J(\overline{\mathbb{Q}(t)}(u)) \cong \mathrm{NS}(S_t)/T$ where T is the trivial lattice spanned by zero section, general fibre and components of reducible fibres of fibration π.

All fibres except the fibre at $u = \infty$ are irreducible. That one looks like this



We find a point $P_0 = (0, t)$ on C_t and we have also a unique point at infinity P_∞. There is a Galois orbit of points

$$P_a = (a, 2a\sqrt{t}(u - t/(2a^2)))$$

where $4a^7 + a^2t^2 - t^3 = 0$.

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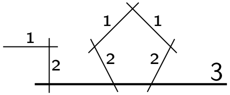

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where $4a^7 + a^2t^2 - t^3 = 0$.

Points on the Jacobian $Q_0 = P_0 - P_\infty$ and $Q_a = P_a - P_\infty$ are linearly independent and form a basis of the Mordell-Weil group (which follows from the height computation and the upper bound). The Néron-Severi lattice is unimodular (because the surface is rational!). From the point count it follows that

$$S_t(\mathbb{F}_q) = 1 + 4q + 2q(1 + \omega(t)^{(q-1)/2}) + qH_q(\alpha, \beta|Mt) + q^2$$

Subgroup of $\mathrm{NS}(S_t)$ spanned by sections P_a (dimension 7) corresponds to the hypergeometric motive.

Number	γ list	Variety	other prop.
1	[* -1, -1, 2 *]	$E([0, \frac{(1+u)^2}{4}, 0, \frac{tu^2(1+u)}{2}, \frac{t^2u^4}{4}])$	rank 1 (A_1^* lattice)

Table: List of degree 1, weight 0 motives

Number	γ list	Variety	other prop.
1	[* 1, -2, -2, -3, 6 *]	$E([-t^3u^2, 1/4t^4(u-1)^2u^4])$	rank=4 (D_4^* lattice)
2	[* -2, -2, 4 *]	D=1, No=1 (non-primitive)	
3	[* -1, -2, 3 *]	$E(0, \frac{u^2}{4}, 0, \frac{1}{2}t(u-1)u^2, \frac{1}{4}t^2(u-1)^2u^2)$	rank = 2 ($A_2^* \oplus Z/3$)

Table: List of degree 2, weight 0 motives

Number	γ list	Variety	other prop.
1	[* -1, 2, -3, -4, 6 *]	$E([tu(u+1), \frac{t^2}{4}])$ (param. 1)	rank=4 (D_4^* lattice)
2	[* -3, -3, 6 *]	D=1, No=1 (non-primitive)	
3	[* -1, -3, 4 *]	$E([0, u^2, 0, 16t(-1+u)^2u, 0])$	rank=3 ($A_3^* + \mathbb{Z}/2$)

Table: List of degree 3, weight 0 motives

Number	γ list	Variety	other prop.
1	[* -1, 2, 3, -4, -6, -6, 12 *]	$E([-u^{2}(u+1)^{2}t, 1/4t^{2}])$	rank =8 (E_8^* lattice)
2	[* 2, -4, -4, -6, 12 *]	D=2, No=1 (non-primitive)	
3	[* 1, -3, -4, -6, 12 *]	$E([-t^{3}(u+1)^{2}u^{2}, t^{5}])$	rank =8 (E_8^* lattice)
4	[* -2, 3, -5, -6, 10 *]	$E([0, t/16, 0, 0, 2t^{4}(u-1)u^{5}])$	rank = 4 (A_4^* lattice)
5	[* 1, -2, -4, -5, 10 *]	$E([0, 4/t+u^{2}, 0, 0, -64u^{5}])$	rank = 4 (A_4^* lattice)
6	[* -1, 3, -4, -6, 8 *]	$E([-tu^{2}(u+1)^{2}, (1/4)t^{2}u^{2}])$	rank =6 (E_6^* lattice)
7	[* -4, -4, 8 *]	D=1, No=1 (non-primitive)	
8	[* 1, -2, -3, -4, 8 *]	$E([-t^{3}u^{2}(u+1)^{2}, t^{5}u^{2}])$	rank =6 (E_6^* lattice)
9	[* -2, -4, 6 *]	D=2, No=3 (non-primitive)	
10	[* -1, -4, 5 *]	$y^{2}=x^{6}+(-2u-2)x^{5}+(u+1)^{2}x^{4}-$ $4tu^{5}$	
11	[* -2, -3, 5 *]	$E([0, 0, 0, -t^{3}u^{3}, \frac{1}{4}t^{4}(u-1)^{2}u^{2}])$	rank=5 (A_5^* lattice)

Table: List of degree 4, weight 0 motives

Number	γ list	Variety	other prop.
1	[* -1, 4, -5, -8, 10 *]	$E([t-u^{3}, -(u^{2}(4t-u^{2}))/4])$	rank =7 (E_7^* lattice)
2	[* -1, 2, -5, -6, 10 *]	$E([0, -tu^{2}, 0, -t^{3}u, t^{4}u(u^{2}+t)]])$	rank=6 (E_6^* lattice)
3	[* -5, -5, 10 *]	D=1, No=1 (non-primitive)	
4	[* 2, -3, -4, -5, 10 *]	$E([0, -t, 0, 0, -\frac{t^{2}(u-1)u^{4}}{1024}])$	rank =5 (D_5^* lattice)
5	[* -1, 2, -4, -5, 8 *]	$E([tu(u+1)^{3}, (1/4)t^{2}(u+1)^{2}]))$	rank = 6 (E_6^* lattice)
6	[* -2, -3, 4, -5, 6 *]	$E([t^{3}u(u+1), (1/4)t^{4}(u+1)^{4}u^{2}])$	rank = 6 (D_6^* lattice)
7	[* -1, -5, 6 *]	$y^{2}=(u+1)^{2}t^{2}x^{6}-4tux+4tu$	

Table: List of degree 5, weight 0 motives

Number	γ list	Variety	other prop.
1	[* -1, 3, 5, -6, -9, -10, 18 *]	dimension 4 elliptic fib.	
2	[* -2, 3, 4, -6, -8, -9, 18 *]	dimension 4 elliptic fib.	
3	[* 3, -6, -6, -9, 18 *]	D=2, No=1 (non-primitive)	
4	[* 1, -4, -6, -9, 18 *]	$E([0, t^2, 0, 16t^4u, 64t^5u^6])$	rank=7, E_7^* lattice
5	[* -2, 5, -7, -10, 14 *]	$y^2 = 4x^5 + 4x^2tu^2 - 4t^2u + t^2$	
6	[* 3, -4, -6, -7, 14 *]	$y^2 = 16t^3x^7 + 4t^2x^4 + 16tu^2x^2 + 16tux$	
7	[* 1, -2, -6, -7, 14 *]	$y^2 = 16tu^2x^6 - 16t^2u - 16t^2x + 4t^2$	
8	[* -3, -4, 5, -10, 12 *]	$E([0, -tu, 0, -t^3, (1/4)t^2u^6 + t^4u]))$	rank=8 (E_8^* lattice)
9	[* 1, -2, -3, -8, 12 *]	$E([-t^3, (1/4)t^4u^2(u+1)^4])$	rank=8 (E_8^* lattice)
10	[* -1, 5, -6, -10, 12 *]	$y^2 = 4tx^5 - 4u^2x - 4u + 1$	
11	[* -2, 4, -6, -8, 12 *]	D=3,No=1 (non-primitive)	
12	[* -6, -6, 12 *]	D=1,No=1 (non-primitive)	
13	[* 1, -2, -5, -6, 12 *]	$E([0, 1, 0, 0, -\frac{64u\left(t-u^2\right)^2}{t^2}])$	rank =6 (D_6^* lattice)
14	[* -1, 3, -6, -8, 12 *]	$E([-t, t(-1+u)^2u^4]),\ u = x_4/x_5$	rank=8 (E_8^* lattice)
15	[* 3, -4, -5, -6, 12 *]	$E([-tu\left(t+u^3\right), \frac{t^2u^4}{4}])$	rank = 7 (E_7^* lattice)
16	[* -2, -3, 4, -8, 9 *]	$E([t^3(-1+u), 1/4t^4(-1+u)^2u^4])$	rank 7 (E_7^* lattice)
17	[* -1, 2, -4, -6, 9 *]	$E([t(-1+u)u^2, 1/4t^2])$	rank 7 (E_7^* lattice)
18	[* -3, -6, 9 *]	D=2,No=3 (non-primitive)	
19	[* 1, -2, -3, -5, 9 *]	$E([-t^3u, 1/4t^4(-1+u)^2u^4])$	rank 7 (E_7^* lattice)
20	[* -2, -6, 8 *]	D=3,No=3 (non-primitive)	
21	[* -1, -6, 7 *]	$y^2 = u^2x^8 + (-2u^2 - 2u)x^7 + (u+1)^2x^6 - 4tu$	
22	[* -2, -5, 7 *]	$y^2 = t^2u^2x^6 + (-4tu + 4t)x + 4tu$	
23	[* -3, -4, 7 *]	$E([-t^3(u-1)^2u, t^5u^5])$	rank=7 (E_7^* lattice)

Table: List of degree 6, weight 0 motives

Thank you!

Application to modular forms

For $t = -1/80$ we have $S = -9$, hence both elliptic curves E_1, E_2 are 2-isogenous over $\mathbb{Q}$. The corresponding modular form for them is an eta product

$$\eta^2(2\tau)\eta^2(10\tau) = q\prod_{n=1}^{\infty}(1-q^{2n})^2(1-q^{10n})^2 = \sum_{n=0}^{\infty} a_n q^n$$

http://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/20/2/1/a/

What we proved is that

$$\operatorname{Tr}\operatorname{Frob}_p \operatorname{Sym}^2 H^1_{et}((E_1)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = H_p(\alpha, \beta|1/t)$$

for $p \nmid 10$. So

$$a_p^2 = p - \frac{1}{p} + \frac{1}{p(p-1)}\sum_{m=0}^{p-2} g(4m)g(-m)^4\omega(-\frac{5}{16})^m$$

Application to modular forms

For $t = -1/4$ we have $S = \sqrt{5}$, hence both elliptic curves E_1, E_2 are 2-isogenous over $\mathbb{Q}(\sqrt{5})$ and defined over $\mathbb{Q}(\sqrt{5})$. The corresponding modular form for them is a Hilbert modular form 2.2.5.1-4096.1-f (`http://www.lmfdb.org/ModularForm/GL2/TotallyReal/2.2.5.1/holomorphic/2.2.5.1-4096.1-f`)

For $p \nmid 10$ we have:

- for $p = \mathfrak{p} \cdot \overline{\mathfrak{p}}$

$$a_{\mathfrak{p}}^2 = p - \frac{1}{p} + \frac{1}{p(p-1)} \sum_{m=0}^{p-2} g(4m) g(-m)^4 \omega(-\frac{1}{64})^m$$

- for p inert

$$\left(\frac{-2}{p}\right) a_p = p - \frac{1}{p} + \frac{1}{p(p-1)} \sum_{m=0}^{p-2} g(4m) g(-m)^4 \omega(-\frac{1}{64})^m$$