Databases of curves

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Why make a database?

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We describe curves as individuals, by means of:

- an equation, like $y^2 = x^3 + ax + b$;
- an automorphic form, like $f = \sum_n a_n q^n$;
- an L-function, like $\sum_n \frac{a_n}{n^s}$; or
- a Galois representation, like $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_\ell)$.

We use our databases to link up the big picture.

So what is in there?

Lots of curves, given by equations. We find these as follows:

In genus 2, every curve X over $\mathbb{Q}$ admits an integral equation

$$X : y^2 + hy = f$$

where the degree of $4f + h^2$ is either 5 or 6.
We use a monomial tree (Kedlaya–Sutherland), to quickly calculate the corresponding discriminant in a large box of coefficients, and pick out the small cases.

We found $66,158$ genus 2 curves of absolute discriminant up to 10^6 (joint work with Booker–Sutherland–Voight–Yasaki). We will see these later while touring the LMFDB.

Linking things up

L-function: linked up under standard conjectures. This determines the conductor of X at the same time (Booker, Dokchitser).

Modular forms: not linked up yet. But we do know where to look (Voight's talk). We get

- a paramodular form if X is typical;
- a classical modular form with quadratic coefficients if X is of GL_2-type over the base;
- a Hilbert or Bianchi modular form if X is of GL_2-type over a quadratic extension;
- a Hecke character if X is CM.

Galois representations: not linked up yet.

More data?

Other invariants:

- the conductor, provably (Bristol and Ulm schools);
- the endomorphism ring of the Jacobian, provably (previous work by Lombardo in genus 2 and general algorithms with Costa–Mascot–Voight, see

  https://github.com/edgarcosta/endomorphisms
- the Sato-Tate group (Harvey–Massierer–Sutherland);
- the Tamagawa numbers at primes of bad reduction (Van Bommel).

This just in

In genus 3, we found

- $67,879$ hyperelliptic curves of discriminant up to 10^7, and
- $82,244$ non-hyperelliptic curves of discriminant up to 10^7.

Their endomorphism rings have been calculated; for this, new algorithms by Molin–Neurohr to compute period matrices were indispensable.

Some statistics

Out of $66,158$ curves of genus 2:

Atypical:	3051
Degree 2 map to EC:	2703
Degree 3 map to EC:	61
Degree 5 map to EC:	17
Degree 7 map to EC:	1
Product of distinct ECs:	2771
Power of an EC:	156
Non-geometrically simple:	125
Geometrically simple:	122

The map of degree 7 arises for the curve 20412.b.734832.1

$$y^2 + (x^2 + x)y = x^6 + 3x^5 + 2x^4 + 7x^3 + 11x^2 + 14$$

which splits over $\mathbb{Q}$ into a product of the ECs 54.a2 and 378.a1.

Some statistics

Out of $67,879$ hyperelliptic curves of genus 3:

Atypical:	2183
Degree 2 map to EC:	2038
Degree 3 map to EC:	65
Degree 5 map to EC:	24
Degree 7 map to EC:	3
Two factors:	2123
Product of distinct ECs:	54
Power of an EC:	0
Non-geometrically simple:	0
Geometrically simple:	6

Of the simple cases, 3 have an automorphism of order 4 over $\mathbb{Q}(\sqrt{-1})$, 1 has RM with discriminant 49, 1 has RM with discriminant 81, and 1 has CM.

Some statistics

Out of 82, 244 non-hyperelliptic curves of genus 3:

Atypical:	3763
Degree 2 map to EC:	3380
Degree 3 map to EC:	127
Degree 5 map to EC:	5
Degree 7 map to EC:	0
Two factors:	3506
Product of distinct ECs:	1
Power of an EC:	5
Non-geometrically simple:	1
Geometrically simple:	12

Of the simple cases, 4 are Picard curves, 7 have RM with discriminant 49, and 1 has RM with discriminant 81. CM was out of reach (but not by much).

Sightseeing

We return to the big picture and take in some sights.