Szpiro’s Conjecture

Conjecture

For each $\epsilon > 0$ there exists a constant $C_\epsilon$ such that if $E$ is an elliptic curve over $\mathbb{Q}$ with minimal discriminant $\Delta$ and conductor $N$, then

$$|\Delta| \leq C_\epsilon N^{6+\epsilon}.$$
Szpiro’s Conjecture

Definition

The Szpiro ratio is

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\[ \sigma = \sigma_E := \frac{\log |\Delta|}{\log(N)}. \]

- Szpiro’s conjecture implies that \( \sigma \) is bounded.
- Szpiro’s conjecture is equivalent to the statement: for all \( M > 6 \) there are only finitely many isomorphism classes of elliptic curves over \( \mathbb{Q} \) such that \( \sigma \geq M \).
Importance of Szpiro’s Conjecture

Szpiro’s conjecture is equivalent to the weak ABC-conjecture.

Let $A, B, C$ be nonzero pairwise coprime integers with $A + B + C = 0$. For each $\epsilon > 0$, there exists a constant $\kappa(\epsilon) > 0$ such that

$$|ABC|^{1/3} < \kappa(\epsilon) N^{1+\epsilon}$$

where $N = \prod_{p | ABC} p$. 

Samuele Anni, Sam Schiavone, Nicholas Triantafillou
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Conjecture

Let $K$ be a number field. There is a constant $c(K) > 0$ such that for all elliptic curves $E/K$ and all non-torsion points $P \in E(K)$,

$$\hat{h}_E(P) \geq c(K) \log(N_{K/Q}(\Delta))$$

where $\hat{h}_E$ is the canonical height on $E$. 
Results In The Literature

- *L’ensemble exceptionnel dans la conjecture de Szpiro*, E. Fouvry, M. Nair, G. Tenenbaum
- *Détermination de courbes elliptiques pour la conjecture de Szpiro*, A. Nitaj
Show that Szpiro’s conjecture holds for “almost all” elliptic curves.
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Measure the density of the set of exceptions.
For $a, b \in \mathbb{Z}$, let $E(a, b)$ be the elliptic curve given by $y^2 = x^3 + ax + b$. 

Theorem

For any $M > 1$, 
$$\lim_{A, B \to \infty} \frac{1}{AB} S_0(A, B; M) = 0.$$
For $a, b \in \mathbb{Z}$, let $E(a, b)$ be the elliptic curve given by $y^2 = x^3 + ax + b$.

Let $S_0(A, B; M)$ be the number of pairs $(a, b)$ such that

$$|a| \leq A, \quad |b| \leq B, \quad \text{and} \quad \sigma_{E(a, b)} \geq M,$$

and such that $\not\exists p$ prime with $p^4 \mid a$ and $p^6 \mid b$. 

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For any $M > 1$,

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Find elliptic curves with large Szpiro ratio.
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Found the curve

\[ E : y^2 + xy = x^3 + x^2 + 349410011109107572x - 775428774618307505842556592 \]

with

\[ \sigma_E = \frac{\log(2^{26} \cdot 3^{52} \cdot 5 \cdot 11^8 \cdot 13 \cdot 19^6 \cdot 31^4)}{\log(2 \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \cdot 31)} \approx 8.811944. \]
Summary of strategy:

1. Find a family of elliptic curves depending on parameters $s, t$ with a torsion point of order $m \in \{2, 3, \ldots, 8\}$.
2. Mod out the subgroup generated by one or more of these torsion points in an attempt to introduce large powers in the discriminant.
3. By solving certain Diophantine equations, determine specific values of the parameters $s, t$ that produce large Szpiro ratios.
4. Apply quadratic twists to try to further increase the Szpiro ratio.
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Generalized Szpiro (Hindry)

For $\varepsilon > 0$, there is a constant $c_\varepsilon$ such that Falting’s height and conductor of any abelian variety $A/\mathbb{Q}$ of dimension $g$ satisfy

$$h_{\text{Falt}}(A) \leq \left(\frac{g}{2} + \varepsilon\right) N_A + c_\varepsilon.$$
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Generalized Szpiro: Hyperelliptic Discriminantant Version

There are constants $c, \kappa$ such that if $C/\mathbb{Q}$ is a hyperelliptic curve of genus $g$, with Jacobian $J$, then $\Delta_C^{\min} \leq c_\varepsilon N_j^{\kappa + \varepsilon}$. 
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Based on analogy with elliptic curves, and a “tentative suggestion” of Lockhart for a related conjecture, we tentatively suggest that $\kappa = 10 = 4g + 2$ might be the right value for genus 2.
Following Nitaj, we look for curves which force the constants in generalized Szpiro to be large.

**Definition: Szpiro Ratio.**

For \( C / \mathbb{Q} \) a hyperelliptic curve with Jacobian \( J \) call

\[
\sigma = \sigma_C = \frac{\log |\Delta_C^{\min}|}{\log N_J}
\]

the Szpiro ratio of \( C \).
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The hyperelliptic discriminant version of generalized Szpiro would imply that for any fixed genus, $\sigma$ is bounded. For many ‘random’ curves we tried, $\sigma$ is between 1 and 3. To ‘test’ the conjectures, let’s look for big $\sigma$. 
Looking for Large Szpiro Ratios

For elliptic curves, wanted large isogeny classes
⇒ started with curves with large torsion.

For hyperelliptic curves, rationale for large torsion is a priori less clear:

1. Even if $J \cong A$, $A$ need not be the Jacobian of a genus 2 curve.
   ▶ $A$ not principally polarized.
   ▶ $A \cong E_1 \times E_2$ (with product polarization) as p.p.a.v.

2. Even if $J(C_1) \cong J(C_2)$, different primes may divide $\Delta_{\text{min}} C_1$ and $\Delta_{\text{min}} C_2$.
   ▶ $C_1$ has bad reduction at $p$ but $J(C_1)$, $C_2$, and $J(C_2)$ have good reduction, and vice versa.

But we want to experiment, and curves with # $J(C)(\mathbb{Q})$ tors. large are interesting, so forge ahead.
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But we want to experiment, and $C$ with $\#J(C)(\mathbb{Q})_{\text{tors.}}$ large are interesting, so forge ahead.
First Experiments

We looked at several families of curves $C_t$ with $\#J(C_t)(\mathbb{Q})$ large generically on the family and several sporadic examples.

- 13-torsion family (Flynn):
  \[ y^2 + (2tx - t)y = -t^2x^2(x - 1)^3 \]

- 15-torsion family (Leprevost):
  \[ y^2 = ((t + 3)x^2 - (2t + 3)x + t + 1)^2 - 4tx^3(x - 1)^2 \]

- 24-torsion family (Howe): See later slide
- Several others.

Howe 24-torsion family had much larger Szpiro ratios than the others.
Constructed by “gluing elliptic curves along 2-torsion”:

\[ F : y^2 = g(x) = x^3 - 31x^2 + 256x \]

\[ E_s : y^2 = f(x) = x^3 + \frac{-8(s^4 + 42s^2 - 147)}{(s^2 + 63)^2} x^2 + \frac{16(s^2 + 7)^3}{(s^2 + 63)^3} \]

\[ J(C_s) \] is the image of \( F \times E_s \) under a \((2, 2)-\)isogeny \( \phi \), where \( \ker(\phi) \) is the graph of an isomorphism of \( F[2] \cong E_s[2] \) as Galois modules.

Equations for \( C_s \) can be given explicitly (and Howe does).
For $s \in \mathbb{Q}$, define

\[
c_4 = -31(s^4 + 42s^2 - (32200/93)s - 147)
\]
\[
c_2 = 2^8(s^8 + 84s^6 - (3472/3)s^5 + 1470s^4 - 48608s^3 + 53508s^2
\]
\[
+ 170128s + 21609)
\]
\[
c_0 = 2^{20}(7/3)s(s^2 + 7)^3(s^2 + 63)
\]
\[
d = s^4 + 42s^2 + (1736/3)s - 147
\]

Then let $C : y^2 = (1/d)(x^6 + c_4x^4 + c_2x^2 + c_0)$. 
1. Conductor (of the Jacobian) is nailed down. $J(C_s) \sim F \times E_s$, so $N_{J(C_s)} = N_F \cdot N_{E_s}$
   ▶ Analogy to searching in isogeny families.
   ▶ Conductor computation is provably correct and much easier.
Heuristic Explanation of Large Szpiro Ratios

1. Conductor (of the Jacobian) is nailed down. \( J(C_s) \sim F \times E_s \), so \( N_{J(C_s)} = N_F \cdot N_{E_s} \)
   - Analogy to searching in isogeny families.
   - Conductor computation is provably correct and much easier.

2. Large ‘extra’ prime factors often appear to high powers (\( \approx 20 \)) in \( \Delta(C_s) \).
   - \( J \) non-simple rules out an obstruction to such primes.
   - If \( J \) has good reduction at \( p \) and \( J_p \) is absolutely simple, then \( C \) has good reduction at \( p \).
\[ s = \frac{1}{12^i}, \quad i = 1, \ldots, 8 \]

| \( \log |\Delta|/ \log(N) \) | \( \approx \) |
|-----------------------------|-------|
| \( \frac{\log 2^{22}3^{6}7^{3}43^{1}211^{1}1009^{3}2042207^{22}}{\log 2^{2}3^{2}7^{2}43^{1}211^{1}1009^{1}} \) | 16.05 |
| \( \frac{\log 2^{28}3^{9}7^{3}23^{1}281^{1}4649^{1}6311^{3}61478548991^{22}}{\log 2^{2}3^{2}7^{2}23^{1}281^{1}4649^{1}6311^{1}} \) | 18.88 |
| \( \frac{\log 2^{34}3^{12}7^{3}317^{1}593429^{1}20901889^{3}1307680847585279^{22}}{\log 2^{2}3^{2}7^{2}317^{1}593429^{1}20901889^{1}} \) | 20.24 |
| \( \frac{\log 2^{40}3^{15}7^{3}19^{12}67^{1}51797^{3}58109^{3}404311147^{1}1430148767862371813^{22}}{\log 2^{2}3^{2}7^{2}67^{1}51797^{1}58109^{1}404311147^{1}} \) | 20.53 |
| \( \frac{\log 2^{46}3^{18}7^{3}3361^{3}6113^{12}15649^{1}128956129^{3}249267937^{1}92189400189327741919^{22}}{\log 2^{2}3^{2}7^{2}3361^{1}15649^{1}128956129^{1}249267937^{1}} \) | 20.28 |
| \( \frac{\log 2^{52}3^{21}7^{3}62412703137793^{1}561714328240129^{1}11686021132862554405802606591^{22}}{\log 2^{2}3^{2}7^{2}62412703137793^{1}561714328240129^{1}} \) | 22.08 |
| \( \frac{\log 2^{58}3^{24}7^{3}41^{22}193^{1}1293^{22}2567447^{1}163237223^{1}8987429251842049^{3}2017161633638299363013881883^{22}}{\log 2^{2}3^{2}7^{2}193^{1}2567447^{1}163237223^{1}8987429251842049^{1}} \) | 22.40 |
| \( \frac{\log 2^{64}3^{27}7^{3}89^{12}179^{12}211^{1}389^{1653^{22}140533^{12}141908506565471^{1}1294189812265254913^{3}343702481742027779629533^{12}}}{\log 2^{2}3^{2}7^{2}211^{1}389^{1}141908506565471^{1}1294189812265254913^{1}} \) | 14.26 |
\[ s = 1/15^i, \ i = 1, \ldots, 8 \]

| \( \log |\Delta|/ \log(N) \) | \( \approx \) |
|-----------------------------|----------|
| \( \log 2^{10}3^65^37^3197^3443^15351^{12} \) | 11.49 |
| \( \log 2^{20}9^67^311^34027^324917^123134463789^{12} \) | 12.30 |
| \( \log 2^{10}3^{12}5^97^311^223^37^1317^31367^35009^1107609^{22}172884889^{12} \) | 14.81 |
| \( \log 2^{19}3^{15}5^17^323^129^1127^322^221493^311827^339254^3123782377^1677190148049^{22} \) | 19.41 |
| \( \log 2^{10}3^{18}5^73^73^7129013^3170473^11513037^3665959^115927025913^{22}63768729341^{12} \) | 16.10 |
| \( \log 2^{21}3^{21}5^73^711^3193^18848351^164537223^344198033^13403520843^1289054921239^3892385549054469031^{22} \) | 17.50 |
| \( \log 2^{19}3^{27}5^73^11^3193^1241177^22411129^128848351^164537223^344198033^13403520843^1289054921239^3892385549054469031^{22} \) | 19.50 |
Next Steps

1. Analyze the effect of taking quadratic twists/experiment with quadratic twists.
   - If $J$ is semisimple, quadratic twisting shouldn’t make Szpiro ratios above 5 larger (up to some possible funny business at 2.)
   - Quadratic twists by primes of good reduction move Szpiro ratio towards 2.5.
   - May be able to analyze additive or mixed reduction in particular families.

2. Consider more families from ‘gluing along torsion.’

3. Better understand when $C$ has bad reduction but $J$ has good reduction and construct large Szpiro examples.

4. Analytic argument a la Fouvry, Nair, Tenenbaum that almost all hyperelliptic curves (ordered by discriminant or coefficients) have Szpiro ratio close to one.
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Thank you!