Exercise. Let $C/\mathbb{Q}_p$, for p odd, be the curve

$$4xy(x+y)=p.$$

Recall that this equation defines a regular model $\mathcal{C}/\mathbb{Z}_p$ of C, and its special fibre $\bar{C}/\mathbb{F}_p$ consists of 3 lines meeting at a point.

(1) Use Thm 2 and point counting on $\bar{C}$ to find the local factor $P(C/\mathbb{Q}_p,T)$.
(2) Use Thm 3 to compute the special fibre $\bar{C}'/\mathbb{F}_p$ of a regular model with normal crossings $\mathcal{C}'/\mathbb{Z}_p$.

[You may use that all f_F and f_L are linear, and so the conditions in Thm 3 are automatically satisfied; alternatively, see below how they are defined. Observe that $\mathcal{C}'$ can be obtained from $\mathcal{C}$ by one blow up at the point of $\bar{C}$ where 3 components meet, and $\mathcal{C}$ from $\mathcal{C}'$ by blowing down the unique component of self-intersection -1.]

How to define the reductions f_F and f_L for $f=\sum_{i,j}a_{ij}x^iy^j$ formally:

Take a 2-face F, and extend $v|_F: F\to\mathbb{R}$ to a unique linear function $v_F:\mathbb{Z}^2\twoheadrightarrow\frac{1}{\delta_F}\mathbb{Z}$ (surjective by definition of δ_F). Pick an identification

$$\alpha:\mathbb{Z}^2 \quad \xrightarrow{\cong} \quad v_F^{-1}(\mathbb{Z}),$$

of affine lattices, and set

$$f_F(x,y)=\sum_{i,j}\frac{a_{\alpha(i,j)}}{\pi^{v_F(\alpha(i,j))}}x^iy^j \qquad \bmod \pi.$$

The definition for $f_L(t)$ is the same, except that $v_L:(\mathbb{Q}\text{-line spanned by } L)\to\frac{1}{\delta_L}\mathbb{Z}$, $\alpha:\mathbb{Z}\cong v_L^{-1}(\mathbb{Z})$ and $f_L(t)$ is univariate. (In the above exercise, it is always linear once the factors of t are removed; t is a unit and has no zeroes in $\mathbb{G}_m$.)

These definitions depend on the choice of α, but different choices give isomorphic reductions $f_F=0\subset\mathbb{G}_m^2$ and $f_L=0\subset\mathbb{G}_m$.