REGULATOR CONSTANTS

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This note is a concise version of [2] §2, and concerns a group-theoretic invariant that owes its existence to the fact that different G-sets can have isomorphic permutation representations.

Let G be a finite group. If $H_i < G$ are subgroups and $n_i \in \mathbb{Z}$, we say that $\Theta = \sum_i n_i H_i$ is a *relation between permutation representations* or simply a *G-relation* if $\bigoplus_i \mathbb{C}[G/H_i]^{\oplus n_i} = 0$ as a virtual representation. (Equivalently, Θ is an element in the kernel of the natural map from the Burnside ring to the representation ring of G.)

Suppose $\Theta = \sum_i n_i H_i$ is such a relation, and ρ is a $\mathbb{Q}G$-representation. Pick a G-invariant non-degenerate $\mathbb{Q}$-bilinear pairing $\langle,\rangle$ on ρ and define the *regulator constant*

$$\mathcal{C}_\Theta(\rho) = \prod_i \det\left(\tfrac{1}{|H_i|}\langle,\rangle|\rho^{H_i}\right)^{n_i} \in \mathbb{Q}^*/\mathbb{Q}^{*2}.$$

Here ρ^{H_i} is the space of H_i-invariant vectors of ρ, and $\det(\langle,\rangle|V)$ is the determinant of the matrix $(\langle v_i, v_j\rangle)_{i,j}$ with v_i any $\mathbb{Q}$-basis of V — as an element of $\mathbb{Q}^*/\mathbb{Q}^{*2}$ it is independent of the basis.

The crucial fact is that $\mathcal{C}_\Theta(\rho)$ is independent of the pairing $\langle,\rangle$. Regulator constants are additive in both Θ and ρ,

$$\mathcal{C}_{\Theta_1+\Theta_2}(\rho) = \mathcal{C}_{\Theta_1}(\rho)\mathcal{C}_{\Theta_2}(\rho), \qquad \mathcal{C}_\Theta(\rho_1 \oplus \rho_2) = \mathcal{C}_\Theta(\rho_1)\mathcal{C}_\Theta(\rho_2).$$

In particular it suffices to describe them for $\mathbb{Q}G$-irreducible representations.

Example 1. Let $G = D_{2p}$ be the dihedral group of order $2p$, with p an odd prime. Its $\mathbb{Q}$-irreducible representations are trivial $\mathbf{1}$, sign ϵ and $(p-1)$-dimensional ρ. There is a unique relation Θ up to multiples, and $\mathcal{C}_\Theta(\mathbf{1}) = \mathcal{C}_\Theta(\epsilon) = \mathcal{C}_\Theta(\rho) = p$. We summarise this as

D_{2p}	$\mathbf{1}$	ϵ	ρ
$\Theta = 1 - 2\,C_2 - C_p + 2\,G$	p	p	p

Example 2. Take $G = A_5$. Here the irreducible rational representations are $\mathbf{1}, \rho, \sigma, \eta$ of dimensions $1, 6, 4$ and 5, respectively, and the subgroups of G up to conjugacy are $1, C_2, C_3, C_2 \times C_2, C_5, S_3, D_{10}, A_4, A_5$. The lattice of relations is generated by 5 elements, and here are the regulator constants:

A_5	$\mathbf{1}$	ρ	σ	η
$\Theta_1 = 1 - 3\,C_2 + 2\,C_2 \times C_2$	2	1	1	2
$\Theta_2 = C_2 \times C_2 - 2\,D_{10} - A_4 + 2\,A_5$	3	1	3	3
$\Theta_3 = S_3 - D_{10} - A_4 + A_5$	3	1	3	3
$\Theta_4 = C_3 - C_5 - 2\,A_4 + 2\,A_5$	15	5	15	3
$\Theta_5 = 1 - 2\,C_2 - C_5 + 2\,D_{10}$	5	5	5	1

Question 3. What are these numbers?

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Let us mention that the field $\mathbb{Q}$ may be replaced by any field K of characteristic 0 (or coprime to $|G|$), provided we stick to self-dual representations: if ρ is a self-dual KG-representation, $\mathcal{C}_\Theta(\rho) \in K^*/K^{*2}$ is well-defined, and is unchanged in field extensions; $\mathcal{C}_\Theta(\rho\otimes_K L)$ agrees with $\mathcal{C}_\Theta(\rho)$ in L^*/L^{*2} for $L\supset K$.

Here are a few cases when we know how to prove that the regulator constants are trivial:

Theorem 4 (Vanishing)**.** *Let ρ be a $\mathbb{Q}G$-irreducible representation, and $\Theta=\sum_i n_iH_i$ a G-relation.*

1. $\operatorname{ord}_p \mathcal{C}_\Theta(\rho)\equiv 0 \mod 2$ *for all $p\nmid |G|$.*
2. $\mathcal{C}_\Theta(\rho)=1$ *if $\rho\otimes\mathbb{C}$ admits a non-degenerate alternating G-invariant pairing. In particular, if a complex irreducible constituent of ρ is non-self-dual, symplectic or has even Schur index.*
3. $\mathcal{C}_\Theta(\rho)=1$ *if ρ is not a constituent of any of the $\mathbb{Q}[G/H_i]$.*
4. $\mathcal{C}_\Theta(\mathbb{Q}[G])=1$*; generally, $\mathcal{C}_\Theta(\mathbb{Q}[G/D])=1$ if $D<G$ is cyclic or has odd order.*
5. $\operatorname{ord}_p \mathcal{C}_\Theta(\mathbb{Q}[G/D])\equiv 0 \mod 2$ *if $D<G$ has a normal subgroup N of order coprime to p and D/N cyclic.*

For the applications that we have in mind, we fix a prime p and to each G-relation Θ associate a representation $\tau_{\Theta,p}$ that encodes the p-part of regulator constants:

$$\tau_{\Theta,p}=\bigoplus_{\substack{\sigma\ \mathbb{Q}G\text{-irr.}\\ \operatorname{ord}_p\mathcal{C}_\Theta(\sigma)\text{ odd}}} (\text{any }\mathbb{C}\text{-irreducible constituent of }\sigma).$$

Its defining property is that for every $\mathbb{Q}G$-representation ρ,

$$\operatorname{ord}_p \mathcal{C}_\Theta(\rho)\equiv\langle\tau_{\Theta,p},\rho\rangle \mod 2.$$

Question 5. For a fixed p, describe which complex representations of G are of the form $\tau_{\Theta,p}$ for some Θ.

Theorem 6. *The set of $\tau_{\Theta,p}$ is closed under direct sum, tensor product by permutation representations, induction and restriction. Moreover,*

1. *$\tau_{\Theta,p}$ has even dimension and trivial determinant.*
2. *If $H<G$, then $\operatorname{Res}^H_G\tau_{\Theta,p}=\tau_{\operatorname{Res}^H_G\Theta,p}$.*
   Here $\operatorname{Res}^H_G(\sum n_iH_i)=\sum n_i\sum_{x\in H\backslash G/H_i} H\cap xH_ix^{-1}$.
3. *If $H<G$ and Φ is an H-relation, then $\operatorname{Ind}^G_H\tau_{\Phi,p}=\tau_{\operatorname{Ind}^G_H\Phi,p}$.*
   Here $\operatorname{Ind}^G_H(\sum n_iH_i)=\sum n_iH_i$.

Remark 7. The name "regulator constant" was introduced in [1] and comes from regulators of elliptic curves. Suppose G is the Galois group of an extension F/K of number fields, and E/K is an elliptic curve. If $\Theta=\sum_i n_iH_i$ is a G-relation, Artin formalism for L-functions forces the identity

$$\prod_i L(E/F^{H_i},s)^{n_i}=1.$$

The Birch–Swinnerton-Dyer conjecture expresses the leading terms of these

L-functions at $s = 1$ in terms of arithmetic invariants of E/F^{H_i}. The above formula leads to

$$\prod_i (\mathrm{Reg}_{E/F^{H_i}})^{n_i} \quad \equiv \quad \text{computable quantity} \quad \mod \mathbb{Q}^{*2}. \tag{8}$$

Here $\mathrm{Reg}_{E/L}$ is the so-called regulator, the determinant of the canonical height pairing $\langle , \rangle_L$ on a basis of $E(L)/\text{torsion}$.

Now consider the $\mathbb{Q}G$-representation $\mathcal{E} = E(F) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $\langle , \rangle = \langle , \rangle_F$ is an $\mathbb{R}$-valued G-invariant $\mathbb{Q}$-bilinear pairing on $\mathcal{E}$, and the determinant $\det(\frac{1}{|H|}\langle , \rangle | \mathcal{E}^H)$ is (up to a rational square) the regulator of E/F^H. So the left-hand side of (8) equals $\mathcal{C}_\Theta(\mathcal{E})$, and hence the parity of $\langle \tau_{\Theta,p}, \mathcal{E} \rangle$ is computable for every Θ and p.

Tables of regulator constants

In the tables, beneath each representation we record its dimension. If the representation is not complex-irreducible, we also write $2o1$, $2n1$ or $1s2$, depending on whether its complex constituent is orthogonal, not self-dual or symplectic. (In each of the examples below, we never have more than 2 constituents.)

$C_2 \times C_2 = C_{2,2}$	ρ_1	ρ_2	ρ_3	ρ_4
	1	1	1	1
$C_1 - C_2^a - C_2^b - C_2^c + 2C_{2,2}$	2	2	2	2
$p = 2$	$*$	$*$	$*$	$*$

Quaternions Q_8	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
	1	1	1	1	4
					$1s2$
$C_2 - C_4^a - C_4^b - C_4^c + 2G$	2	2	2	2	1
$p = 2$	$*$	$*$	$*$	$*$	

D_8	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
	1	1	1	1	2
$C_2^b - C_2^c + C_{2,2}^a - C_{2,2}^b$	1	2	1	2	2
$C_2^a - C_4 - C_{2,2}^a - C_{2,2}^b + 2G$	2	2	2	2	1
$C_1 - C_2^a - 2C_2^c + 2C_{2,2}^a$	2	2	2	2	1
$p = 2$		$*$		$*$	$*$
	$*$	$*$	$*$	$*$	

D_{16}	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
	1	1	1	1	2	4
						$2o1$
$C_2^b - C_2^c + D_8^a - D_8^b$	1	1	1	1	2	2
$C_{2,2}^a - C_{2,2}^b - D_8^a + D_8^b$	1	1	2	2	2	1
$C_4 - C_8 - D_8^a - D_8^b + 2G$	2	2	2	2	1	1
$C_1 - 2C_2^c - C_4 + 2D_8^a$	1	1	1	1	1	1
$C_2^a - C_4 - 2C_{2,2}^b + 2D_8^b$	2	2	2	2	1	1
$p = 2$					$*$	$*$
			$*$	$*$	$*$	
	$*$	$*$	$*$	$*$		

$G_{20} = C_5 \rtimes C_4$	ρ_1	ρ_2	ρ_3	ρ_4
	1	1	2	4
			$2n1$	
$C_2 - 2C_4 - D_{10} + 2G$	5	5	1	5
$C_1 - 2C_2 - C_5 + 2D_{10}$	5	5	1	5
$p = 5$	$*$	$*$		$*$

$\mathrm{SL}_2(\mathbb{F}_3)$	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
	1	2	4	4	3
		$2n1$	$1s2$	$2n1$	
$C_4 - C_6 - Q_8 + G$	2	1	1	1	2
$C_2 - 3C_4 + 2Q_8$	2	1	1	1	2
$p = 2$	$*$				$*$

$\mathrm{GL}_2(\mathbb{F}_3)$	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7
	1	1	2	4	3	3	4
				$2n1$			
$S_3^a - S_3^b$	1	1	1	1	1	1	3
$C_4 - C_{2,2} + D_8 - Q_8$	1	2	2	1	1	2	1
$D_8 - D_{12} - Sy_{16} + G$	2	1	2	1	2	1	1
$C_{2,2} - C_6 - D_8 + SL_2(\mathbb{F}_3)$	2	1	2	1	2	1	1
$C_2^a - C_{2,2} - C_6 - C_8 + 2G$	6	3	6	1	2	1	1
$C_{2,2} - C_8 - 2D_8 + 2Sy_{16}$	2	1	2	1	2	1	1
$C_3 - C_6 - 2S_3^b + 2D_{12}$	2	2	1	1	2	2	3
$2D_8 - Q_8 - 2D_{12} + SL_2(\mathbb{F}_3)$	3	3	3	1	1	1	1
$C_1 - 2C_2^b - C_6 + 2D_{12}$	6	6	3	1	2	2	1
$p = 2$		$*$	$*$			$*$	
	$*$		$*$		$*$		
$p = 3$							$*$
	$*$	$*$	$*$				

$\mathrm{PSL}_2(\mathbb{F}_7)$	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
	1	6	6	7	8
		$2n1$			
$A_4^a - A_4^b$	1	1	2	1	1
$S_4^a - S_4^b$	1	1	2	1	1
$C_{2,2}^a - C_{2,2}^b$	1	1	2	1	1
$A_4^a - H_{21}^1 - S_4^b + G$	1	1	1	1	1
$C_{2,2}^a - S_3 - D_8 + S_4^b$	2	1	1	2	1
$C_2 - C_3 - C_{2,2}^a + A_4^a$	2	1	2	2	1
$S_3 - D_8 - H_{21}^1 + G$	6	1	2	6	3
$C_1 - C_2 - C_3 - C_4 + 2S_4^a$	6	1	2	6	3
$C_3 - C_4 - 2A_4^a + 2S_4^b$	3	1	2	3	3
$C_{2,2}^a - C_7 - 3A_4^b + 3H_{21}^1$	1	1	1	1	1
$p = 2$			$*$		
	$*$			$*$	
$p = 3$	$*$			$*$	$*$

S_3	ρ_1	ρ_2	ρ_3
	1	1	2
$C_1 - 2C_2 - C_3 + 2G$	3	3	3
$p = 3$	$*$	$*$	$*$

S_4	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
	1	1	2	3	3
$C_{2,2}^b - S_3 - D_8 + G$	2	1	2	1	2
$C_2^b - C_3 - C_{2,2}^b + A_4$	2	1	2	1	2
$C_2^a - C_2^b - C_{2,2}^a + C_{2,2}^b$	1	2	2	2	1
$C_1 - 2C_2^a - C_3 + C_{2,2}^a + A_4$	1	1	1	1	1
$C_2^b - C_4 - 2C_{2,2}^b + 2D_8$	2	1	2	1	2
$C_{2,2}^a - 2C_{2,2}^b + 2S_3 - A_4$	3	3	3	1	1
$p = 2$	$*$		$*$		$*$
		$*$	$*$	$*$	
$p = 3$	$*$	$*$	$*$		

S_5	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7
	1	1	4	4	5	5	6
$D_8 - D_{12} - G_{20} + G$	1	1	1	1	1	1	1
$C_2^a - C_3 - C_{2,2}^b + A_4$	2	1	1	1	1	2	2
$C_{2,2}^b - S_3^b - D_8 + S_4$	2	1	1	1	1	2	2
$C_2^b - C_3 - C_{2,2}^a + A_4$	2	2	1	1	2	2	1
$S_3^a - D_8 - A_4 + S_4$	6	2	1	3	2	6	1
$D_8 - D_{10} - S_4 + A_5$	2	6	3	1	6	2	1
$C_{2,2}^a - 2S_3^a + A_4$	3	3	3	3	3	3	1
$C_2^a - C_{2,2}^b - C_5 - S_3^a + D_{10} + A_5$	10	5	5	5	1	2	10
$C_2^a - 2C_{2,2}^b - C_6 + 2D_{12}$	3	1	1	3	1	3	1
$C_4 - C_6 - 2G_{20} + 2G$	6	1	1	3	1	6	2
$C_1 - 3C_2^b + 2C_{2,2}^a$	2	2	1	1	2	2	1
$C_4 - C_{2,2}^b - S_3^a + 2D_8 - 2G_{20} + A_5$	10	10	5	5	2	2	5
$p = 2$	$*$					$*$	$*$
	$*$	$*$			$*$	$*$	
$p = 3$	$*$			$*$		$*$	
		$*$	$*$		$*$		
$p = 5$	$*$	$*$	$*$	$*$			$*$

A_4	ρ_1	ρ_2	ρ_3
	1	2	3
		$2n1$	
$C_2 - C_3 - C_{2,2} + G$	2	1	2
$C_1 - 3C_2 + 2C_{2,2}$	2	1	2
$p = 2$	*		*

A_5	ρ_1	ρ_2	ρ_3	ρ_4
	1	6	4	5
		$2o1$		
$C_2 - C_3 - C_{2,2} + A_4$	2	1	1	2
$S_3 - D_{10} - A_4 + G$	3	1	3	3
$C_{2,2} - 2S_3 + A_4$	3	1	3	3
$C_1 - 3C_2 + 2C_{2,2}$	2	1	1	2
$C_3 - C_5 - 2A_4 + 2G$	15	5	15	3
$p = 2$	*			*
$p = 3$	*		*	*
$p = 5$	*	*	*	

A_6	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
	1	5	5	16	9	10
				$2o1$		
$C_{2,2}^a - C_{2,2}^b$	1	2	2	1	1	2
$S_3^a - S_3^b + S_4^a - S_4^b$	1	2	2	1	1	2
$D_8 - D_{10} - H_{36}^9 + G$	2	2	2	1	2	1
$D_8 - D_{10} - S_4^b + A_5^a$	2	2	2	1	2	1
$D_8 - D_{10} - S_4^a + A_5^b$	2	2	2	1	2	1
$S_3^b - D_{10} - A_4^a + A_5^b$	3	1	3	1	1	3
$S_3^a - D_{10} - A_4^b + A_5^a$	3	3	1	1	1	3
$C_{3,3} - A_4^a - A_4^b + H_{18}^4$	2	3	3	1	2	2
$H_{18}^4 - S_4^b - A_5^b + G$	2	6	6	1	2	1
$C_2 - C_4 - C_{2,2}^a - S_3^a + D_8 + S_4^b$	1	1	1	1	1	1
$C_{2,2}^a - 2S_3^b + A_4^a$	3	1	3	1	1	3
$C_4 - C_5 - S_4^b - H_{36}^9 + A_5^a + G$	5	2	2	5	5	2
$C_3^a - C_4 - C_{3,3} + S_4^a - A_5^a + G$	3	2	6	1	1	6
$C_1 - C_2 - C_3^a - C_4 + 2S_4^b$	6	3	1	1	2	6
$C_3^b - C_4 - C_{3,3} - H_{18}^4 + 2S_4^b$	6	1	3	1	2	6
$C_4 - 2S_3^b + 2H_{18}^4 - H_{36}^9 - A_5^a + A_5^b$	1	6	6	1	1	2
$p = 2$		*	*			*
	*	*	*		*	
$p = 3$	*		*			*
	*	*				*
$p = 5$	*			*	*	

M_{11}	ρ_1 1	ρ_2 10	ρ_3 20 $2n1$	ρ_4 11	ρ_5 32 $2n1$	ρ_6 44	ρ_7 45	ρ_8 55
$H_{36}^9 - H_{36}^9$	1	1	1	1	1	1	1	1
$S_3^a - S_3^b$	1	3	1	3	1	3	1	3
$A_5^a - A_5^b$	1	3	1	3	1	3	1	3
$D_8 - D_{12} - G_{20} + S_5$	1	1	1	1	1	1	1	1
$C_4 - C_6 - Q_8 + SL_2(\mathbb{F}_3)$	2	1	1	2	1	1	2	2
$C_8 - Q_8 - H_{72}^{39} + H_{72}^{41}$	1	1	1	1	1	1	1	1
$GL_2(\mathbb{F}_3) - H_{72}^{39} - S_5 + H_{720}$	1	1	1	1	1	1	1	1
$A_4 - H_{18}^3 - S_4 + H_{72}^{40}$	2	1	1	2	1	1	2	2
$C_6 - Q_8 - H_{18}^3 + H_{72}^{41}$	3	1	1	3	1	1	1	1
$S_3^b - Q_8 - H_{18}^4 + H_{72}^{41}$	3	1	1	3	1	1	1	1
$C_3 - C_4 - C_{3,3} + H_{36}^9$	3	3	1	1	1	3	1	3
$H_{72}^{40} - S_5 - H_{144}^{182} + H_{720}$	3	1	1	3	1	1	1	1
$H_{36}^9 - A_5^a - H_{72}^{41} + A_6$	3	2	1	3	1	1	2	2
$H_{36}^9 - H_{36}^{10} + H_{72}^{40} - H_{72}^{41}$	1	2	1	1	1	1	2	2
$A_6 - H_{660} - H_{720} + G$	6	2	1	6	1	1	1	1
$H_{18}^3 - SL_2(\mathbb{F}_3) - H_{36}^9 + H_{72}^{41}$	6	1	1	6	1	1	2	2
$C_{3,3} - A_4 - SL_2(\mathbb{F}_3) + H_{72}^{40}$	1	6	1	3	1	3	2	6
$H_{36}^9 - GL_2(\mathbb{F}_3) - H_{72}^{41} + H_{144}^{182}$	6	1	1	6	1	1	2	2
$H_{18}^4 - 2S_4 + H_{36}^9$	2	1	1	2	1	1	2	2
$S_5 - H_{144}^{182} - H_{660} + G$	10	2	1	10	1	5	1	1
$D_{12} - Sy_{16} - H_{36}^9 + H_{72}^{41} - S_5 + H_{720}$	1	2	1	1	1	1	2	2
$G_{20} - H_{36}^9 - H_{55}^1 - S_5 + A_6 + H_{660}$	5	1	1	5	1	5	1	1
$S_4 - 2H_{36}^9 + H_{72}^{41}$	3	6	1	1	1	3	2	6
$Q_8 - Sy_{16} - G_{20} - SL_2(\mathbb{F}_3) + GL_2(\mathbb{F}_3) + S_5$	6	2	1	6	1	1	1	1
$C_{2,2} - D_8 - Q_8 - H_{36}^9 + 2H_{72}^{41}$	1	2	1	1	1	1	2	2
$C_1 - C_2 - C_3 - C_4 + 2S_4$	6	3	1	2	1	3	2	6
$C_2 - C_4 - C_8 - D_{10} - H_{18}^3 + GL_2(\mathbb{F}_3) + H_{144}^{182} + A_6$	6	1	1	6	1	1	2	2
$C_5 - C_6 - G_{20} + H_{36}^9 - H_{55}^1 + H_{144}^{182} + H_{660} - H_{720}$	10	1	1	10	1	5	2	2
$Sy_{16} - G_{20} - H_{36}^9 + H_{72}^{41} + H_{660} - G$	30	1	1	30	1	5	2	2
$D_{10} - Sy_{16} - G_{20} + H_{72}^{41} - S_5 + H_{144}^{182}$	30	2	1	30	1	5	1	1
$C_4 - C_5 - C_{11} + S_4 + H_{36}^9 + H_{55}^1 - A_5^b - 2H_{72}^{40} + H_{660} - 3H_{720} + 3G$	6	2	1	6	1	1	1	1
$p = 2$	*			*			*	*
		*					*	*
$p = 3$		*		*		*		*
	*			*				
$p = 5$	*			*		*		

References

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[2] T. Dokchitser, V. Dokchitser, Regulator constants and the parity conjecture, 2007, arxiv: 0709.2852.

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