Cayley Graphs

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Abstract

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INTRODUCTION

In mathematics it is often a continues process: starting from a very abstract object, e.g. group, which as such is difficult to grasp, mathematicians introduced first multiplication tables and then CG's to visualise them, and finally, the extreme product is a particular picture, e.g. drawn on the paper or by a computer. This is already beyond mathematics - pictures are not mathematical abstract objects but physical concrete representation of them. The things in between groups and pictures of CG's are intermediate objects - they may be more and more intuitive and visualisable representations of abstract mathematical objects but they still preserve the initial structure. Indeed, one can recover a group from a CG; moreover, it is possible to do it using a picture of a CG. Because there is no a big gap between the stages: the group and the CG, and the CG and the picture, i.e. the syntax of for the 'notation' (drawing) of a CG is very strict the former is quite reliable.

Main QUESTIONS TO ANSWER: -what is the epistemic role of Cayley graphs? (epistemic justification and support, illustration, suggestion, etc.) -in virtue of which properties they play this role? -what is the mathematical status of CG's? -what are the main ontological manifestations of CG's? -in those epistemic roles are CG's more advanced in revealing the structure of groups than other representations of the groups playing similar roles, e.g. group tables? -is the diagrammatic thinking with CG's is sufficient for a proof? can one substitute a part of a proof in terms of groups by the part of a proof in terms of Cayley graphs? (this means CG's are replaceable, they are logically superfluous, but a proof by CG's is cognitively more comprehensible. They provide a different epistemological view on the objects.)

1 Introduction to Cayley graphs

1.1 General Overview

Cayley graph is a colored graph, which was introduced by Cayley in 1878 as a graphic representation of abstract groups. A group is a set of elements together with an operation that combines any two of its elements to form a third element. The set and operation must satisfy group axioms, namely associativity, identity and inverse elements. Cayley graph depicts the elements of the groups as vertexes connected by colored edges. Every colour corresponds to a generator (a fixed element which can generate all the other elements). While the groups are widely employed in various mathematical areas and even such diverse applied fields as quantum theory and crystallogrphy, the formulation of the axioms is detached from the concrete nature of the group and its operation. This abstract character of the axiomatic representation of groups allows one to handle entities of very different mathematical origins in a flexible way, while retaining essential structural aspects of many objects in abstract algebra and beyond. Cayley graphs preserve this abstraction but at the same time make it accessible for study the structure of groups and mathematical manipulation over them.

1.2 Official Definition

Let us say this in a more formal way. Def. 1. A group is a set, G, together with an operation "$\circ$" that combines any two elements a and b to form another element denoted $a \circ b$. To qualify as a group, the set and operation, $(G, \circ)$, must satisfy four requirements known as the group axioms:

1. Closure

   For all a, b in G, the result of the operation $a \circ b$ is also in G.

2. Associativity

   For all a, b and c in G, the equation $(a \circ b) \circ c = a \circ (b \circ c)$ holds.

3. Identity element

   There exists an element I in G, such that for all elements a in G, the equation $I \circ a = a \circ I = a$ holds.

4. Inverse element

   For each a in G, there exists an element b in G such that $a \circ b = b \circ a = I$, where I is the identity element.

Example 1. One of the most familiar examples of groups is the set of integers Z over addition. A sum of any integers is an integer, therefore it belongs to the group; the equation $a + 0 = a$ identifies 0 is an identity element, and the equivalence $a + b = b + a = 0$ implies that any element a has an inverse element $-a$ in the group. **Example 2.** Another simple example is a group of rotation of an equilateral triangle: consider the motions of equilateral triangle which rotates in its plane about an axis through its center. The groups elements will be those motions of the triangle which bring the triangle into coincidence with itself (*congruent* motions). To determine these motions we need to fix one of the vertices of the triangle, let's call it A. For a triangle to coincide with itself it is not necessary to require each individual (labeled) vertex coincide with itself, but only that the set of points forming the triangle coincide itself with the set of points in the initial position. For example, the rotation to $2/3\pi$ will give a desirable motion or $\pm 2k\pi$.

Let us now define Cayley graphs. Def2. Let G be a group, I an identity element of G and S a generating set of G (such that I is not in S). Then the Cayley graph $CG(G, S)$ is the following directed graph:

1. $CG = \langle V, E \rangle$
2. $V = G$
3. For any x,y in V: $\langle x, y \rangle$ is in E, i.e. there is s in S such that $y = xs$

In this context, such S is called a set of generators.

This was one of possible definitions, there are different in different books. Here is one, which reflects the coloring of generators:

Let G be a group ($G = \langle X, \circ \rangle$), let S be a generating set of G, such that the identity element of G is not a member of S. Then $CG = \langle V, E, c \rangle$ is the Cayley graph that is determined by G and S:

1. $V = G$
2. E is a set of oriented edges between members of V, i.e., $E \subset V \times V$
3. $c : E \to S$
4. $\forall x, y \in V : \langle x, y \rangle \in E \Leftrightarrow y = x \circ c(\langle x, y \rangle)$

For our first example, $G = Z$, a Cayley graph in respect to the set S consisting of the standard generator 1 and its inverse (-1 in the additive notation) is an infinite chain (see th picture below).

For our second example, the Cayley graph of an equilateral triangle rotation group can be depicted as a triangle with one way directed edges.

PIC

A graph can be drawn in many ways, as the only thing that is important is which nodes are connected to which. It does not matter how the nodes are arranged on the picture, and while the edges are usually drawn as straight lines, but this is not a necessity. The edges may cross, as long as it is clear that this crossing point is not another node. Below there are some different ways to draw the same graph.

1.3 Fundamental Properties of Cayley Graphs: Group-Graph Correspondences

There are rules how to construct a graph for a given group and a given set of generators. These rules are, at the same time, the keys for recovering a group for a given Cayley graph. Group elements correspond to CG's vertexes, the chosen generators correspond to colors (if there is only one fixed generator there is one (default) colour), 'words' correspond to paths, word for the identity element (e.g. $rrr = r3 = I$) corresponds to a closed path (a path for which initial and final points coincide), solvability of equation $rx = s$ corresponds to the fact that the network is connected (there is a path from any point to any other).

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2 CG's and proofs: Banach-Tarski Paradox

"The Banach-Tarski paradox is a theorem in set theoretic geometry which states that a solid ball in 3-dimensional space can be split into several non-overlapping pieces, which can then be put back together in a different way to yield two identical copies of the original ball (only moving the pieces around and rotating them, without changing their shape). The reason the Banach-Tarski theorem is called a paradox is because it contradicts basic geometric intuition. "Doubling the ball" by dividing it into parts and moving them around by rotations and translations, without any stretching, bending, or adding new points, seems to be impossible, since all these operations preserve the volume, but the volume is doubled in the end.

Unlike most theorems in geometry, this result depends in a critical way on the axiom of choice in set theory. This axiom allows for the construction of nonmeasurable sets, collections of points that do not have a volume in the ordinary sense and require an uncountably infinite number of arbitrary choices to specify.

Here we sketch a proof which is similar but not identical to that given by Banach and Tarski. Essentially, the paradoxical decomposition of the ball is achieved in four steps:

1. Find a paradoxical decomposition of the free group in two generators.
2. Find a group of rotations in 3-d space isomorphic to the free group in two generators.
3. Use the paradoxical decomposition of that group and the axiom of choice to produce a paradoxical decomposition of the hollow unit sphere.
4. Extend this decomposition of the sphere to a decomposition of the solid unit ball.

We now discuss the first step, on which a Cayley graph is applied, in detail, skipping the rest of the proof.

Step 1. The free group with two generators a and b consists of all finite strings that can be formed from the four symbols a, a^{-1}, b and b^{-1} such that no a appears directly next to an $a^{-1)}$ and no b appears directly next to a b^{-1}. Two such strings can be concatenated and converted into a string of this type by repeatedly replacing the "forbidden" substrings with the empty string. For instance: $abab^{-1}a^{-1}$ concatenated with $abab^{-1}a$ yields $abab^{-1}a^{-1}abab^{-1}a$, which contains the substring $a^{-1}a$, and so gets reduced to $abaab^{-1}a$ One can check that the set of those strings with this operation forms a group with neutral element the empty string e. We will call this group $F2$. The sets $S(a^{-1})$ and $aS(a^{-1})$ in the Cayley graph of $F2$.

The group $F2$ can be "paradoxically decomposed" as follows: let $S(a)$ be the set of all strings that start with a and define $S(a^{-1})$, $S(b)$ and $S(b^{-1})$ similarly. Clearly,

$$F_2 = \{e\} \cup S(a) \cup S(a^{-1}) \cup S(b) \cup S(b^{-1})$$

but also

$$F_2 = aS(a^{-1}) \cup S(a)$$

and

$$F_2 = bS(b^{-1}) \cup S(b)$$

The notation $aS(a^{-1})$ means take all the strings in $S(a^{-1})$ and concatenate them on the left with a. By these partitions we "shift" some of them by multiplying with a or b, then "reassemble" two of them to make $F2$ and reassemble the other two to make another copy of $F2$. This is exactly how we want to decompose the ball.

3 Philosophical Reflections: Ontology

We have completed the technical part of the introduction of CG's; now we are moving to philosophical reflections about CG's. In this section we analyse ontological questions concerning CG's: their ontological status in respect to groups, their geometrical nature and how it manifests itself in different ontological dimensions of CG's.

3.1 *Sui Generis* Objects

Cayley graphs are not just useful heuristic tools for visualising groups, they are mathematical objects in their own right. But at the same time they are the same as group *cum* letters, group *cum* generating set. Logically, they are the same but they are distinct in terms of mathematical practise: they allow extra manipulations on them, use of geometrical/topological language in addition to algebraic language (magnitudes, growth, vertexes, etc). Also, there is an epistemic distinction: we have conceptual access to group structure explained algebraically, whereas we have perceptual access to group structure and generating set by means of the realisation of a CG's.

The fact that CG's are mathematically independent objects is reflected in the presence of mathematical problems (conjectures, theorems, proofs) about CG's themselves. For example, there is a CG's formulation of the Lovàsz conjecture (1970):*Every finite connected Cayley graph contains a Hamiltonian cycle.* The same conjecture but formulated in graph theory terms says: *Every finite connected vertex-transitive graph contains a Hamiltonian path.* The advantage of the Cayley graph formulation is that such graphs correspond to a finite group G and a generating set S. Thus one can ask for which G and S the conjecture holds rather than attack it in full generality. Although CG's are properties of groups, this version of the conjecture speaks about CG's themselves and one cannot say they are just a heuristic aid in this case. Moreover, this example demonstrates how CG's can contribute both to graph and group theory in virtue of their intermediate nature: they provided a solution for a graph theory problem and reveal new aspects of group theory (having a Hamiltonian CG is a property of the group).Therefore, in some contexts CG's can be identified with groups, in some there are reasons to see them as distinct objects.

3.2 Geometrical Objects

In which sense are CG's and groups are "geometrical" objects? When does the "geometrical" start? As soon as you talk about spacial objects like spaces, lines, planes, dots? As soon as you specify a metric? **Are Cayley graphs geometrical?**. The definition of a Cayley graph is $CG = \langle V, E \rangle$. This is an abstract mathematical object. *Prima facie*, there is nothing strictly suggesting that it has geometrical properties, neither does the notion of group. But as soon as we consider V and E as "vertexes" and "edges", we give it a geometrical essence. The question now is whether it has this property of geometricality simply because it may be visualised as a concrete picture. However, almost anything can be visualised somehow even if the "somehow" in question is rather bizarre; but this facility of theirs is not sufficient to bestow the property of geometricality. For an object to have geometricality requires to have some attributes of a natural geometrical object, such as line, triangle, circle.

CG's are geometrical in the sense that their pictures involve some geometrical components: dots and edges. But at the same time, the length and position of them does not matter, what matters is just their connection, therefore the abstract structure. Nevertheless, there is an evident difference between a picture of a Cayley graph and, for example, a picture of a triangle: at least, (1) a CG may be directed and (2) it may be coloured. A line considered as an abstract geometrical object can be visualized by means of drawing; whereas the vertexes and edges of a Cayley graph are not representations of abstract objects of geometry, but of the elements and generators of a group.

Therefore, there is a sense in which CG's are geometrical objects, and also a sense in which they are not. CG's are geometrical in the sense that their realizations can be embedded in plane. They are non-geometrical in the sense that their realizations depict groups with a fixed set of generators, which are not objects of geometry. In other words, they are a subset of a Euclidean plane (taking this latter as $R \times R$) but they are constructed in a way external to Euclidean geometry: they are obtained from groups by transformations which are not geometrical - the transformation is one of a group into a graph, but graphs are not the objects of geometry as mathematical discipline; graphs, in general, belongs neither to geometry, no algebra.

Are Groups geometrical?. Are groups geometrical objects? "Groups were first studied as symmetries of geometric objects, and later as fundamental groups of topological space. That is, one starts by considering a geometric or topological object and the group is determined by this object.

In the 1960s and 1970s mathematicians began to consider groups as geometric objects themselves, and not as something that is defined by another geometric or topological object. Lie groups for instance are considered as differentiable manifolds, and finitely generated groups are considered as Cayley graphs. Some geometric properties of finitely generated Cayley graphs can be considered as properties of the group itself. These are mainly properties which are quasi-isometry invariants like the number of ends, growth, hyperbolicity, accessibility and many more. The task of geometric group theory is to relate such geometric properties with algebraic properties of the group. In the 1980ies and 1990ies geometric group theory has become an own branch of mathematics which today is gaining more and more importance."

Group with a set of generators is a group with a metric, is the same as a Cayley graph.

3.3 Five dimensions

Cayley graphs, along with algebraic groups, are abstract mathematical objects. They represent the group properties with respect to chosen generators. Groups have an abstract structure, which initially does not imply any geometrical properties - they are sets and binary operations of certain type. Cayley graphs provide geometrical representation of the structure of the groups, but this is accomplished by means of different "dimensions" of Cayley graphs. We distinguish five dimensions of Cayley graphs: *groups with a fixed set of generators*, *Cayley graphs*, *geometrical representations* (figures) of CG's (groups) and *physical realisations* (pictures). Let us call the first two "algebraic" and the second two "geometrical". In order to answer the main problem of the paper, namely, what is the epistemic role of CGs, it is useful to understand what is their ontological status and structure in respect to how they are used in mathematical practise. Let me consider these five dimensions in details.

3.3.1 Groups

First, a group G with the fixed set of generators S is an algebraic object. From a practical point of view, of a geometric group theorist, it is isomorphic or logically equivalent to a correspondent Cayley graph of G with respect to S; from a philosophical point of view, we would say that group is not a graph and they are different objects - Cayley graphs are *sui generis.* Example?

3.3.2 Graphs

Second, a Cayley graph $\langle V, E\rangle$ in respect to S, which, by definition, is not yet a geometrical object. We can also say that $\langle V, E\rangle$ is an abstract mathematical object, which is defined along with the group, in an axiomatic way. At this appearance of CGs there is a linguistic intention from a set-theoretic language in the definition of group (set, elements, binary operations) toward a language which embraces "geometrical" terms (vertexes, edges, path, closed path). Nevertheless, formally, it is still an algebraic representation. But it has already a disposition for a "geometrical embodiment" of $\langle V, E\rangle$, it allows for thinking of CGs as 2 or 3 dimensional figures. Incidentally, a group with a fixed set of generators can be considered as a group with a fixed metric, which gives a presupposition for seeing it as a geometrical object. In a sense, at the first and second stages, as Cayley graphs as groups are formalised using geometrical terms, nevertheless are algebraic objects. This official algebraic axiomatic formalisation is very important for incorporation of the objects in the formal theory and preserving an algebraic access to these objects. Generally speaking, it is a broadly employed practise to make mathematical objects accessible for different mathematical languages/theories.

3.3.3 Figures

Third, CG's can be represented or thought in a geometrical way, as objects with spatial characteristics or figures. The elements of sets V and E may be interpreted in a geometrical way as connected dots - vertexes and edges. This "geometrical embodiment" is a peculiar one because it gives rise to new links between a given group and elements of other mathematical theories. For example, edges of the graph situated in a coordinate system give a numerical interpretation, in real or natural numbers, and this in turn can be connected with set-theoretical interpretation. Can a group be directly represented in a geometrical way? It seems to be unclear how. At least it is difficult to figure out what could be a unified way to do it. Cayley graphs fill this gap. And crucially, an application of graphs instead of some other types of diagrams brings the graph theory in service. This geometrical manifestation allows for the next dimension of CG's - a concrete visualisation of a CG a picture on the paper.

3.3.4 Pictures

Finally, the geometrical representation may be implemented in multiple particular pictures, which are physical and concrete. From one view, pictures have a special status in mathematics: they are distinct from mathematical objects; in a sense, they are syntactic notations. As we need a formula to be visualised

????by means of symbols in some media, e.g. on the paper, we need a graph to be drawn. Indeed, there is quite a leap between a CG as an abstract object and a picture. One need to know how to depict generic 'sintactic' elements, which in the case of CG's are dots and arrows. From this view, pictures are open to different interpretations, which is correct; it is our interpretation what directs the reasoning with a picture in question. From the point of view we are employing in this paper, pictures are one of manifestations of an abstract object, this approach seems to be more generic as it explains particular features of a picture in question and eliminates irrelevant interpretations.

4 Epistemology

4.1 Epistemic motivation for CG's

Originally groups were intended to describe the symmetries of geometric objects and permutations, which are also intuitive concepts. Then groups were defined by axioms as abstract objects. Finally, they were considered as geometric objects themselves, i.e. Cayley graphs, still defined axiomatically. The link between algebraic and geometric properties of groups is the main purpose of geometrical group theory, and CG's are the key element.

The short review of the history of development of geometric group theory methodology reflects two main interests: go along with intuitions and hold axiomatic framework. CG's are the result of this methodology. They were motivated by the epistemic need of an intuitive and, at the same time, in theoretic representation of groups.

Groups themselves alike the Kantian concepts are in a sense "empty": without an object on which they can act their action is unclear, they are just sets with a particular operation on them. For groups to exhibit their structure, to be "alive", means to act on some spaces. CG's are good recipients in this sense; similar to the light reflected from objects they illuminate how groups function.

For an infinite but finitely generated group, the geometry of the Cayley graph is independent of choice of finite set of generators, hence an intrinsic property of the group. (This is only interesting for infinite groups: every finite group is coarsely equivalent to a point (or the trivial group), since one can choose as finite set of generators the entire group.) Formally, for a given choice of generators, one has the word metric (the natural distance on the Cayley graph), which determines a metric space.

Geometric group theory is a mathematical area devoted to the study of *finitely generated* groups by exploring the connections between *algebraic* properties of such groups and *topological and geometric properties of spaces* on which these groups act (that is, when the groups in question are realised as geometric symmetries or continuous transformations of some spaces). The geometric group theory considers *finitely generated* groups themselves as geometric objects. This is usually done by studying the Cayley graphs of groups, which, in addition to the graph structure, are endowed with the structure of a metric space, *given by* *the so-called word metric.*

We see at least three main functions of CG's: illustrative/heuristic application: geometry is 'closer' to physical phenomena pragmatic: enriching the tool-kit employing other areas of mathematics

4.2 Epistemic Roles of the Five Dimensions

Following our approach of five manifestations dimensions of CG's we now move to the analysis of their epistemic roles.

The role of the first dimension - algebraic-axiomatic - is to provide an abstract formal definition, which incorporates an object to a formal theory. The formal $CG = \langle V, E \rangle$ definition of CG's gives a precise correspondence with a given group in respect to the set of generators. It gives us a definite knowledge what CG's are.

The second dimension gives the idea of an abstract object with geometrical intention. The third is to allow geometrical thinking about the object: thinking it as a geometrical figure with spacial structure. Finally,**Pictures** allow to perceive the geometrical attributes like shape or colour and do some geometrical constructions on them. As Cayley graphs are accessible for geometrical embodiment, they in turn are accessible for depicting and, therefore, perceiving and working for visual reasoning. Moreover, they consent to further imaginative manipulations like thought experiments, metaphorical thinking and so on which can provide further insights about mathematical relations and properties. Here is an example: The distance between two puzzle positions is the least number of moves you need to get from one to the other. Similarly, the distance between two nodes in a (Cayley) graph is the least number of edges you have to traverse to get from one node to the other. In general this is a rather hard problem because there are so many paths/move-sequences that are possible, and not only must they connect the two nodes/positions, they must be the shortest possible. In the simple cases of the uniform polyhedra it is easy to find distances just by inspecting the shape. You can just follow the edges that go to the end node in as straight a line as possible, and it will be a shortest path. Without such visual cues, this is hard.

Suppose you have a (Cayley) graph made out of string - each node is a knot, and the edges are strings of all the same length. This would look a bit like one of those net bags that oranges are sold in. If you hold it at one node and let it dangle, then all the adjacent nodes will hang at the same height just below the top node. All nodes at distance two will hang at a level just below that, nodes at distance three below that, and so on. The node(s) furthest away will hang furthest down.

In general graphs the diameter is defined as the distance between two nodes that are furthest apart. Cayley graphs are uniform, so this dangling string net will look exactly the same regardless of which node you picked it up at. In particular, the diameter is the distance between the top and bottom of this dangling net.

Let's push this dangling net bag visualisation a bit further. When you pick it up, most of the material hangs as a bunch near the bottom. It is like a teardrop shape. This observation matches what is shown by calculations of permutation puzzles. The number of puzzle positions increases by a fairly steady factor every step further away from the start position, until most puzzle positions have been reached. After this the number of positions further out decreases drastically within a few steps. The distribution of positions at each distance is similar to the thickness of the net bag at each height. On the right you see an example, which is same graph as the cuboctahedron in example 2 above (except that two horizontal edges have been removed for clarity). In this case the diameter is 3, and there is only one antipode, one position at the furthest distance.

4.3 Application to Banach-Tarski Theorem Proof

The role of Cayley graph in BTP proof: More than an Illustrative Role From the first glance, The CG plays an illustrative role in the proof. Nevertheless, note that even reasoning about the group decomposition, i.e. about the patrician formulas, does involve a number of algebraic notions, which have a geometrical origin (partition, shift, union). This suggests a question, whether the decomposition can be done with a Cayley graph in virtue of its geometrical features. If it is the case, CG may play some epistemological role (justification or support) in the discovery of the proof, namely, leading to the idea of the proof. It is easy to see how the decomposition is possible reasoning just by means of the Cayley graph. The subsets $S(a)$, $S(a^{-1})$, $S(b)$ and $S(b^{-1})$ of the decomposition of the group are the main graph branches and one can see from the picture that the Cayley graph of F_2 consists of four branches....... one other four branches, the left one, $S(a^{-1})$, is just shifted by one step to the right (being multiplied by a from the left) and united with the right branch, $S(a)$. At the same time the shifted branch is equal to three branches: itself united with upper and down branches. With the Cayley graph the paradoxical decomposition becomes vivid. Also, the picture shows the fractal structure of the graph, which explains the "paradoxality" of the decomposition: the fractal structure of the graph suggests that all the set of vertexes and edges is actually present at each point of the graph (of four branches), therefore there are infinitely many four-branch decompositions. it leads to the idea that the procedure of the paradoxical decomposition can give infinitely many segments of the required type. This property of the Cayley graph is given transparently only by the picture but not by the analytical representation.

This last observation suggests that the use of the diagram involves some dynamics: the imaginary construction of the graph at each point. It is rather construction in imagination - it is enough just notice the fractal structure and then make imagine the same picture at each point.

4.4 Implications to the Concept of Proof

practically speaking ARGUMENT: 1. Whatever is a component of a mathematical theory T is, practically speaking, epistemologically indispensable to T (up to subsitutivity). 2. CG's are, practically speaking, epistemologically indispensable to Geometrical Group Theory. 3. CG's are developed in order to supplement the axiomatic definition of groups with a geometrical heuristic, which enhances intuitive access to group structure. Therefore, intuitions are, practically speaking, epistemologically indispensable to Geometrical Group Theory. I apprehend "geometrical" in the sense discussed above. This sense of geometry is the area of intuition. Therefore, intuition is the core of the goal of the theory. Therefore, the role of intuition is important.

About the use of "heuristic": this word is usually used for some device or method or technique which helps one make a discovery (that P) but which provides no epistemic justification or support (evidence or reason) for the belief that P. As epistemologists are mainly concerned with epistemic justification and support, they have not been much concerned with heuristic tools. One kind of example of a heuristic device is a diagram which suggests that an object has a certain kind of symmetry, which, if the symmetry were proved, would enable one to solve some problem. Another kind of example would be a diagram which gives one a central idea for a proof of a theorem without itself being part of a proof.

two questions: (1) What exactly is the relation of logical equivalence between pieces of reasoning? Usually we think of logical equivalence as a relation between propositions. (2) How can one establish that a piece of diagrammatic reasoning is logically equivalent (in the relevant sense) to a piece of ordinary reasoning?

From Tim R. CG's can be in principle eliminated from proofs and replaced by terminology of groups but in a very superficial way - they are properties of groups, why do we need to eliminate them. do we have to eliminate the pictures? are they part of reasoning? it is not pictures what constitutes a proof but our reasoning with them. pictures even if they suggestive they are strongly suggestive: one has to look at the picture in order to figure out the proof. one need to see the shape of CG. then it becomes natural to talk in terms of CG's. It is possible to translate all the talk into groups terms but it will be rather superficial. exactly because a group with metric is a CG.

4.5 Intuition and Visualization: Programmatic Remarks

What can we learn about intuition from this example?

1. Mathematicians need intuitions to have an access to abstract mathematical structures in order to go on in their investigations.

2. The nature of these intuitions is "geometrical" in a sense (connected to spacial, imaginary and visual)

3. There is a blend of algebraic and geometric elements in this nature. This fact is reflected in the language geometric algebraists use, which combines

algebraic and geometric terms (e.g., in considering a Cayley graph as "a set of vertexes" or treating Cayley graphs as magnitudes and exploring its "growth").

4. Intuitions are largely employed in higher' levels of abstractions along with elementary mathematics.

5. Intuitions can be accommodated in mathematical theory by introducing more intuitive objects, like Cayley graphs.

6. Intuitions evolve along with mathematical research: immediate (evident) and superficial intuitions like symmetries of geometrical objects, receive their formal-theoretical reflections by axiomatization and later were enriched by more sophisticated intuitions implemented in Cayley graphs.

Fallibility Proofs by means of CG's are no more fallible then their algebraic analogues. They are diagrams - graphs but they are precise in their construction

IT IS NOT JUST A PICTURE what PROVES BUT IT IS OUR WORK ON THIS PICTURE.

5 conclusion

summery CG's are *sui generis* mathematical objects in contrast to the view they are just heuristic tools to understand the structure of groups. CG's were originally developed for 'geometrical' visualizational purpose are 'geometrical' in this sense but not in the sense as belonging to geometry. In fact, they have different dimensions (I-IV). Each dimension has different aspects of 'geometricality': I - groups came from symmetries, but they are algebraic, II - 'vertexes' and 'edges' are geometrical terms, III-CG's allow geometrical embeddiment (or embedding?) in figures, IV - can be depicted and perceived. Finally, 'geometricality' was attributed to groups themselves. Each dimension has its epistemic function.