MECH1 **Problem Sheet 4** **Due Friday March 13**

Force impulse. Work and energy for one degree of freedom systems.

1. Explain qualitatively the results of the following experiment. Two identical cardboard rings are hung vertically at the same height on two separate stands placed at a distance of some 50cm from each other on the table. One takes a solid wooden stick of length some 60cm and diameter of some 1cm and puts its two ends through the two rings, so that the stick is positioned horizontally above the table.

   If one pulls down at the stick, gradually increasing the effort, one or both rings will break. However, if one administers a sharp blow to the centre of the stick instead, then the stick will break, while the cardboard rings remain intact. Why?

   Answer: In the former scenario, the system can be viewed as in equilibrium at all times. Hence, the net tension in the rings is equal $mg + F$, where F is the value of the pulling force. If the rings can withstand tension not higher than some T_0, then one will break as soon as $mg + F > 2T_0$.

   Before the rings break they have to be stretched, however. In order to stretch the rings one needs to set the *ends* of the stick (and of course the rest of the stick between the ends and the centre where the blow is administered) in motion. This requires some force impulse, and no matter how strong is the force involved in the blow, if the blow takes a very short time, then the force impulse, and hence the momentum change of the stick over this short time will be too small to sufficiently stretch the rings to have them they break. So, throughout the short time of the interaction one can basically assume that the ends of the stick are not moving, i.e. the centre of the stick takes the whole brunt of the impact.

2. A brick of mass m slides from rest down the incline of the angle α and length L. The friction coefficient is $\mu < \tan\alpha$. After the brick reaches the bottom of the incline it continues moving horizontally with the same friction coefficient until it stops. Calculate the net work of the friction force as (i) the change of the brick's energy, and (ii) by definition of work; verify that you get the same answer.

   Answer: (i) The change of the brick's energy (as it begins at rest at the altitude $h = L\sin\alpha$ and ends at rest at zero altitude) is $mgL\sin\alpha$. This should equal the work of the friction force.

   (ii) Indeed, the work of the friction force equals $\mu mg\cos\alpha L$ over the time as the brick moves down the incline, plus μmgx, where x is the distance it travels horizontally before it stops. The horizontal motion occurs with the constant acceleration $-\mu g$ and initial velocity $v_0 = \sqrt{2L[g(\sin\alpha - \mu\cos\alpha)]}$. (Indeed, the expression in the square brackets is the constant acceleration while moving down the incline. Recall that moving a distance d from rest with constant acceleration a means the terminal velocity is $\sqrt{2da}$.) So, we should have $v_0 = \sqrt{2[\mu g]x}$, in other words

   $$x = L(\mu^{-1}\sin\alpha - \cos\alpha).$$

   So, the total work done by the friction force equals

   $$A = \mu mg\cos\alpha L + \mu mgx = mgL\sin\alpha,$$

   as it should by (ii).

3. A heavy closed chain of mass M is hanging over a massless block which can rotate without friction. A monkey of mass m uses the chain as a vertical treadmill: it professes to be climbing up the chain, yet the monkey's height above the floor remains the same at all times. If the monkey is capable of producing maximum power W, how soon after having started the exercise will it have to give up?

   Answer: The monkey pushes the chain down with some vertical force F arising from the monkey's interaction with the chain. By Newton's third law and second laws, $F = mg$, because the monkey is at rest with respect to the ground. Hence, the chain will move with constant acceleration $a = \frac{m}{M}g$ and if the exercise takes time t, a given point on the chain will travel the distance $s = at^2/2$. This is all due to the monkey's work, with the force mg. Hence, the work and power

   $$A = mgs = (mgt)^2/2M, \qquad W = dA/dt = (mg)^2 t/M.$$

If $W_{\max}$ is the maximum power the monkey is capable of producing, it will give up then after $t = MW_{\max}/(mg)^2$.

4. Two springs with constants k_1 and k_2 have the same length and are attached to the ceiling. A light rod is hung on the two springs, so that the springs are parallel to each other and the rod is parallel to the ground. A heavy weight of mass M is now attached to the rod, so that the rod remains parallel to the ground after the springs have stretched and the system is in equilibrium. Find the ratio of the springs' potential energies.

   Answer: $U = kx^2/2$, and as the springs are stretched by the same x, $U_1/U_2 = k_1/k_2$, assuming that the unstretched potential energy is zero for both springs.

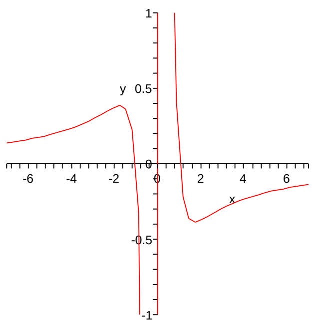
   A body of mass M is hung vertically on a spring with constant k_2, whose loose end is attached to another spring with constant k_1. The loose end of the latter spring is attached to the ceiling. The system is in equilibrium. Find the ratio of the springs' potential energies.

   Answer: If T_1 is the tension of the first (upper) spring and T_2 is the tension of the second – lower – spring, then $T_1 + T_2 = Mg$, in addition to this, $T_1 = T_2$, because the only force pulling at the upper spring comes from the deformation of the lower spring. So, the springs' deformations $x_1/x_2 = k_2/k_1$, then the potential energies

$$\frac{U_1}{U_2} = \frac{k_1 x_1^2}{k_2 x_2^2} = \frac{k_2}{k_1}.$$

5. The potential energy function for a particle of mass m is given as $U(x) = \frac{1}{x^3} - \frac{1}{x}$. Sketch U, identify and describe equilibrium points and regions of finite/infinite motion. Find the period of small oscillations near the stable equilibrium, if any.

   Answer: $U(x)$ is odd, so the graph is symmetric w.r.t. origin. For positive x, if $x \to 0$, then the first term is much greater, so $U \to +\infty$. For $x \to +\infty$ the second term is greater in absolute value, so $U(x)$ for large x goes to zero and is negative. Hence, there must be a minimum somewhere for $x > 0$, and indeed the only critical point for $x > 0$ is at $x = \sqrt{3}$, where $U' = 0$. The value of U there is $-\frac{2}{3\sqrt{3}}$. This is a stable equilibrium, with energy $E = -\frac{2}{3\sqrt{3}}$. The second derivative at the equilibrium $k = U''(\sqrt{3}) = \frac{2}{3\sqrt{3}}$. The period of small oscillations is therefore $T = 2\pi\sqrt{\frac{k}{m}}$, with k as above.



   Similarly, there is an unstable equilibrium at $x = -\sqrt{3}$, with energy $E = \frac{2}{3\sqrt{3}}$.

   Since $E = K(\dot{x}) + U(x)$, and $K(\dot{x}) = \frac{m\dot{x}^2}{2} \geq 0$, the particle cannot possibly have energy E and be positioned at x such that $U(x) > E$. E.g., the particle cannot have energy $E < -\frac{2}{3\sqrt{3}}$ and be positioned anywhere at $x \geq 0$.

   Finite motions: for $x > 0$ and $-\frac{2}{3\sqrt{3}} \leq E < 0$: the particle will move periodically between the two values of $x > 0$ which solve $\frac{1}{x^3} - \frac{1}{x} = E$. For $x < 0$ and $E \leq \frac{2}{3\sqrt{3}}$ the particle will move between the greatest $x < 0$ solving $\frac{1}{x^3} - \frac{1}{x} = E$ and $x = 0$, reaching infinite speed as $x \to -0$.

   Infinite motions: for $x > 0$ and $E \geq 0$, starting at $x > 0$ solving $\frac{1}{x^3} - \frac{1}{x} = E$. For $x < 0$ and $0 < E \leq \frac{2}{3\sqrt{3}}$, beginning at the smallest $x < 0$ solving $\frac{1}{x^3} - \frac{1}{x} = E$. For $x < 0$ and $E > \frac{2}{3\sqrt{3}}$, beginning at $x = 0_-$, where the speed is infinite.