MECH1 **Problem Sheet 5 Solutions**

Work and Energy, continued

Do all problems

1. (This is the last problem from the previous homework) The one-dimensional potential energy function for a particle is given as $U(x) = \frac{1}{x^3} - \frac{1}{x}$. Sketch U, identify and describe equilibrium points and regions of finite/infinite motion.

   See the previous homework solutions (on http://www.maths.bris.ac.uk/∼maxmr/Mechanics1/sol4.pdf)

2. Given $\boldsymbol{F}(x, y) = xy\boldsymbol{i} + \frac{y}{x}\boldsymbol{j}$ compute the work of $\boldsymbol{F}$ along the path $y = x^2$, between the origin and the point $(1, 1)$.

   Answer: Use x as the parameter for the curve:$x = t$. Along the curve $x = x, y = x^2$ $d\boldsymbol{r} = dx(1, 2x)$, the force is $\boldsymbol{F}(x) = (x^3, x)$. Hence $\boldsymbol{F} \cdot d\boldsymbol{r} = (x^3 + 2x^2)dx$, and work

$$A = \int_0^1 (x^3 + 2x^2)dx = 1/4 + 2/3 = 11/12.$$

   The force is not potential. Indeed, suppose for some $U(x, y)$:

$$xy = \frac{\partial U}{\partial x}, \quad \frac{y}{x} = \frac{\partial U}{\partial y}.$$

   Integrating the first equation: $U = x^2y/2 + C(y)$. Taking the partial with respect to y yields $x^2/2 + C'(y)$. This must be equal to $\frac{y}{x}$. So $C'(y) = \frac{y}{x} - x^2/2$. We have a function of y in the left-hand side and a function of (x, y) in the right-hand side of the equality. This is an absurd statement.

3. A particle is moving in three dimensions along the right helix of radius R and pitch (the vertical distance between nearby loops) H. Given $\boldsymbol{F}(x, y, z) = y\boldsymbol{i} + x\boldsymbol{j} + xyz\boldsymbol{k}$ compute the work of $\boldsymbol{F}$ along the part of the helix that starts at $(R, 0, 0)$ and ends at $(-R, 0, H/2)$.

   Answer: The corresponding part of the spiral is $x(t) = R\cos t,\ y(t) = R\sin t,\ z(t) = Ht/2\pi$, with $t \in (0, \pi)$. Hence $dr = dt(-R\sin t, R\cos t, H/2\pi)$ and

$$A = \int_0^\pi dt(-R^2\sin^2 t + R^2\cos^2 t + R^2\frac{H^2 t}{(2\pi)^2}\sin t\cos t) = \int_0^\pi dt(R^2\cos 2t + R^2\frac{H^2 t}{8\pi^2}\sin 2t) = -R^2\frac{H^2}{16\pi}.$$

   Indeed, the first term is zero, as $\cos 2t$ is π-periodic; after integration of the second term by parts the only nonzero term is $-R^2\frac{H^2 t}{16\pi^2}\cos 2t|_0^\pi$.

   The force is not potential. Indeed, suppose for some $U(x, y, x)$:

$$y = \frac{\partial U}{\partial x}, \quad x = \frac{\partial U}{\partial y}, \quad xyz = \frac{\partial U}{\partial z}.$$

   The first two equations imply $U = xy + C(z)$, for some function $C(z)$, and $C'(z)$ is then to be $\frac{\partial U}{\partial z}$ – clearly not equal to xyz.

4. Suppose a force $\boldsymbol{F}(\boldsymbol{r})$ is central. I.e. $\boldsymbol{F}(\boldsymbol{r}) = f(r)\frac{\boldsymbol{r}}{r}$, where r is the distance form the origin. Show that such a force is potential (i) by definition of work, showing that the integral involved between any points A and B depends only on the distances $r(A)$ and $r(B)$ from the origin, but not on the particular path connecting A and B; (ii) by demonstrating that, in fact, if $U(r)$ is such that $U'(r) = f(r)$, then $\boldsymbol{F}(\boldsymbol{r}) = \nabla u$.

   Then find the potential corresponding to the gravitational interaction force

$$\boldsymbol{F}(\boldsymbol{r}) = -G\frac{m_1 m_2}{r^2}\frac{\boldsymbol{r}}{r}.$$

Answer: The elementary work done by $\boldsymbol{F}$ is $dA = f(r)\frac{\boldsymbol{r}\cdot d\boldsymbol{r}}{r}$. But $\boldsymbol{r} \cdot d\boldsymbol{r} = rdr$, hence $dA = f(r)dr$, and

$$A_{12} = \int_{r_1}^{r_2} f(r)dr = U(r_1) - U(r_2), \qquad \text{with } \ U(r) = -\int f(r)dr.$$

Conversely, take $U = U(r) = U(\sqrt{\boldsymbol{r}\cdot\boldsymbol{r}})$ and compute its partial derivatives. E.g.

$$\frac{\partial U}{\partial x} = U'(r)\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = U'(r)\frac{x}{r},$$

similar for $\frac{\partial U}{\partial y}$, $\frac{\partial U}{\partial z}$. Thus

$$\nabla U = U'(r)\,\frac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{r} = U'(r)\hat{\boldsymbol{r}},$$

where $\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}$. For instance, the gravity force

$$\boldsymbol{F} = -G\frac{m_1 m_2}{r^2}\hat{\boldsymbol{r}},$$

the potential is

$$U(r) = Gm_1m_2\int\frac{1}{r^2} = -G\frac{m_1m_2}{r} + C.$$

(In the assignment the minus sign in front of $\boldsymbol{F}$ was forgotten.)

5. A particle of mass m_1 moves with constant velocity v_1 and hits the resting particle of mass m_2. After that the two particles stick together and continue to move as a single particle of mass $m_1 + m_2$. Use conservation of momentum to show that the collision resulted in the loss of kinetic energy. Calculate this loss. Discuss where the mechanical energy could go.

   Answer: After the collision, the compound particle moves with the velocity $v = m_1v_1/(m_1+m_2)$, due to the conservation of momentum. The change in the kinetic energy is therefore

   $$\Delta K = \frac{(m_1+m_2)v^2}{2} - \frac{m_1v_1^2}{2} = -\frac{\mu v_1^2}{2}, \quad \text{where} \quad \mu = \frac{m_1m_2}{m_1+m_2}.$$

   So, *Mechanical* energy here is not conserved – it has decreased. The reason is that the kinetic energy of the particles at the moment when they collide gets transferred into their *internal* energy, resulting in the increase of their temperature, which represents the inner average motions of the molecules forming the particles. Analysing temperature is beyond the expertise of Classical mechanics. If one takes the change in the internal energy before and after the collision into account, the *total physical* energy of the system is, of course, preserved.

6. A ballistic pendulum is a heavy sack filled with sand of mass M, suspended on a string of length l. A bullet of mass m traveling horizontally hits the sack and gets stuck in it, whereupon the sack starts oscillating with angular amplitude α. Find the velocity of the bullet before it hits the sack.

   Answer: The collision of the bullet and the sack is non-elastic. However, the momentum is preserved, and the horizontal speed of the system after the collision is $V = \frac{mv}{M+m}$. By the time the ballistic pendulum reaches the turning point when the angle becomes α all the kinetic energy

   $$K = \frac{(M+m)V^2}{2}$$

   will transfer into the potential $(M+m)gl(1-\cos\alpha)$. Thus

   $$v = \frac{m+M}{m}\sqrt{2gl(1-\cos\alpha)}.$$