MECH1 **Problem Sheet 6 Solutions**

Constants of motion, angular momentum

1. The net force on a particle moving in three dimensions is $\boldsymbol{F} = a\boldsymbol{i} + b\boldsymbol{j}$ with $a, b \neq 0$. Find the direction(s) in which (i) the momentum, (ii) angular momentum are conserved.

   Answer: Since $\dot{\boldsymbol{p}} = \boldsymbol{F}$, the momentum is preserved in any direction $\boldsymbol{d}$ such that $\boldsymbol{F} \cdot \boldsymbol{d} = 0$. (Meaning that the projection of the momentum on the direction $\boldsymbol{d}$ is constant). I.e, for any $\boldsymbol{d}$ lying in the plane, spanned by the z-direction (vector $\boldsymbol{k}$) and the vector $(-b, a, 0)$.

   Since $\dot{\boldsymbol{L}} = \boldsymbol{r} \times \boldsymbol{F}$, the angular momentum is preserved in any direction $\boldsymbol{d}$ such that the projection of torque $\boldsymbol{r} \times \boldsymbol{F}$ on this direction is zero. This is clearly the direction of $\boldsymbol{F}$ itself ($\boldsymbol{r} \times \boldsymbol{F}$ is perpendicular to $\boldsymbol{F}$), and, in fact, no other direction, in the sense that $\boldsymbol{d} \cdot (\boldsymbol{r} \times \boldsymbol{F}) = 0$ for all $\boldsymbol{r}$ only for $\boldsymbol{d}$ collinear with $\boldsymbol{F}$.

2. A bead of mass m whirls on the frictionless table, held to circular motion by a string that passes through a hole in the centre of the table. The string is *slowly* pulled through the hole, so that at every moment t the bead can be thought to be in the state of circular motion, with radius r and angular velocity $\omega(r)$. Originally $r = r_0$ and $\omega = \omega_0$. Find the dependence of the bead's kinetic energy on r and verify that its change equals the work of the force $T(r)$ impelled by the rope. HINT: How does *slowly* enable one to express the latter force as a function of r?

   Answer: The force acting on the bead is central, so the angular momentum is preserved and equals $L = mr_0(\omega_0 r_0)$ in magnitude, from the initial conditions. And whenever r is the distance to the origin,

$$r^2\omega(r) = \omega_0 r_0^2, \qquad \omega(r) = \omega_0 \frac{r_0^2}{r^2}.$$

   The kinetic energy

$$K(r) = mv^2/2 = m(\omega(r)r)^2/2 = m(\omega(r)r)^2/2 = m\omega_0^2 r_0^2 \frac{r_0^2}{2r^2}.$$

   The change in kinetic energy is then

$$K(r) - K(r_0) = m\omega_0^2 r_0^2 \left(\frac{r_0^2}{2r^2} - \frac{1}{2} \right).$$

   On the other hand, as the rope is pulled in slowly, at every time t one may say, for the tension $T(r)$ of the rope that

$$T(r) = m(\omega(r)^2 r) = m\frac{(\omega_0 r_0^2)^2}{r^3},$$

   which is mass times acceleration. In vector terms,

$$\boldsymbol{T} = -m\frac{(\omega_0 r_0^2)^2}{r^3}\frac{\boldsymbol{r}}{r},$$

   i.e. the central force is directed to the origin.

   So, since $\boldsymbol{r} \cdot d\boldsymbol{r} = rdr$ (see the notes, p.31) the work of T between $r = r_0$ and $r = r$ equals

$$-m\omega_0^2 r_0^4 \int_{r_0}^{r} \frac{dr}{r^3} = \frac{1}{2} m\omega_0^2 r_0^4 \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right),$$

   which indeed matches the above expression for the kinetic energy change.

Central force field, polar coordinates

1. Can a repelling central force result in circular orbits?

   Answer: No: the equation for r is $m\ddot{r} = f(r) + \frac{L^2}{mr^3}$. The second term is the centrifugal force. If f is repelling, i.e. $f(r) > 0$ for all r, then $\ddot{r} > 0$ and therefore the circualr orbit solution $r(t) = const$, which implies $\ddot{r} = 0$ is impossible.

2. A particle moves under a cental force $\boldsymbol{F}(\boldsymbol{r}) = -Kr^4\hat{\boldsymbol{r}}$, with $K > 0$. Is this an attractive or repelling force? Find the radius and energy of circular orbit(s), if any. Can a repelling central force result in circular orbits?

   Answer: Attractive, because of the minus sign, acts towards the origin. Circular orbit: when the effective force is zero:

   $$0 = -Kr^4 + \frac{L^2}{mr^3}, \qquad r = \sqrt[7]{\frac{L^2}{Km}}.$$

   The energy of the circular orbit is

   $$E = mv^2/2 = m(\omega r)^2/2 = m\left(\frac{L}{mr^2}r\right)^2/2 = \frac{L^2}{2mr^2},$$

   with r given by the seventh root above. Recall that $\omega = \dot{\theta} = \frac{L}{mr^2}$, so if $r(t) = const$, ω is also constant.

   We can eliminate L from $r = \sqrt[7]{\frac{L^2}{Km}}$ and $\omega = \dot{\theta} = \frac{L}{mr^2}$ to obtain

   $$Kr^3 = m\omega^2,$$

   this relation is necessary to be obeyed by the initial conditions $r(0)$ and $\dot{\theta}(0)$ in order that the particle move along the circular orbit.

   Another necessary condition, of course, is $\dot{r}(0) = 0$. Indeed, if the particle is on the circular orbit, then $r = const.$, so $\dot{r}(t) = 0$ for all t, in particular $t = 0$.

3. A particle of mass m moves in the central force field with the force function $f(r) = -Kr^3$, with $K > 0$. Sketch the effective potential, and argue that all the orbits are bounded. Find the radius and period of circular orbit(s), if any.

   Answer: Similar to the above problem, the circular orbit has radius

   $$r = \sqrt[6]{\frac{L^2}{Km}}.$$

   The period is

   $$T = 2\pi/\omega = \frac{2\pi}{\frac{L}{mr^2}},$$

   r given above.

   The effective potential is the integral of the effective force:

   $$U_{eff} = Kr^4/4 + \frac{L^2}{2mr^2}.$$

   This function clearly goes to $+\infty$ for r going to zero and infinity, with a single minimum, corresponding to the above circular orbit. Hence, r is finite for every orbit (as for the finite total energy E one must always have $E \geq U_{eff}$.

4. A particle of mass m moves in the central force field with the force function $f(r) = -K/r^3$, with $K > 0$. Find the nonzero values of the angular momentum L when circular orbits are possible. For these particular values of L, describe the (non-circular) orbits, given that initially the particle's velocity had some non-zero radial component.

   Answer: In this case the effective force is

   $$-K/r^3 + \frac{L^2}{mr^3},$$

   and it is identically zero as long as

   $$L^2 = Km.$$

   So, if $L^2 = Km$, there exist circular orbits of any radius, and the sign of L simply determines the direction of rotation along these orbits.

For other values of L the effective force is either always positive or always negative (which corresponds to $r(t)$ always strictly increasing or decreasing) and therefore there are no circular orbits – the latter require the effective force to be zero.

Returning to the case $L^2 = Km$ – in other words, the initial position r_0 and the initial $\dot{\theta}(0) = \omega_0$ are such that

$$L^2 = Km = (mr_0^2\omega_0)^2,$$

the equations of motion in polar coordinates are then

$$\dot{r} = 0, \qquad \dot{\theta} = \frac{L}{mr^2}.$$

So, in general, $r(t) = At + B$, and the circular orbit is when $A = 0$. In both cases $B = r_0$, while $A = \dot{r}(0)$. (Clearly, being on a circular orbit requires a special initial condition $\dot{r}(0) = 0$.)

If $A \neq 0$, then, depending on the sign of A, r increases or decreases, while

$$\theta(t) = \theta_0 + \int_0^t \frac{L}{m} \frac{d\tau}{(At + r_0)^2} = \theta_0 + \frac{L}{Am} \left(\frac{1}{r_0} - \frac{1}{At + r_0} \right)$$

So when t changes from 0 to infinity, the total change of θ will be $\frac{1}{r_0}$, i.e the particle will go to infinity (provided that $A > 0$) having made the total of $\frac{1}{2\pi r_0}$ revolutions around the origin.

If $A < 0$, the particle falls onto the centre in a finite time $t = r_0/|A|$.