MECH1 **Problem Sheet 7** **Due Monday May 11**

Oscillations

Do all problems. Terminology: the damped harmonic oscillator equation is

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0,$$

if the right-hand side is not zero, it is referred to as forcing. Always assume that $\omega \gg \gamma$.

1. (This is a revision-type problem: consider it in the mass centre frame.) Find a period of oscillations of a system that consists of two masses m and M, connected by a massless spring with constant k. Answer: $2\pi\sqrt{\frac{\mu}{k}}$, with $\mu = \frac{mM}{m+M}$.

   Answer: $2\pi\sqrt{\frac{\mu}{k}}$, with $\mu = \frac{mM}{m+M}$.

   Indeed, let x, X be the deviations of masses from equilibrium, both pointing in the same direction. We have, by the Second and Third laws

   $$m\ddot{x} = -k(x - X), \qquad M\ddot{X} = k(x - X).$$

   Dividing the first equation by m, the second by M, and then subtracting the second one from the first one yields

   $$\ddot{\chi} = -\frac{k}{M^{-1} + m^{-1}}\chi, \qquad \chi = x - X.$$

   This is a simple harmonic oscillator equation with $\omega^2 = \frac{k}{\mu}$, and hence the period as stated.

2. A mass m can slide without friction on a horizontal plane, being attached to a vertical wall via two consecutive springs with constants k and K. Find the period of oscillations. Answer: $2\pi\sqrt{\frac{m}{\kappa}}$, with $\kappa = \frac{kK}{k+K}$. (This is also a revision-type problem:argue that the net force acting on the point where the springs are connected must be zero.)

   Describe what would change in your solution if the mass was hanging vertically.

   Answer: Indeed, let x, X be the deformation of the two springs, both positive if the springs are stretched. The deviation of the mass from equilibrium is $x + X$, the force acting on it comes from the second spring only:

   $$m(\ddot{x} + \ddot{X}) = -kx.$$

   Besides, at the point where the springs are connected the net force must be zero, or a small mass put therein would move with acceleration going to infinity. Hence, $kx = KX$, i.e. if one of the springs stretched, so must be the other. Eliminating X yields

   $$m(1 + \frac{k}{K})\ddot{x} = -kx,$$

   i.e. $m\ddot{x} = -\kappa x$, a simple harmonic oscillator with frequency $\omega^2 = \frac{\kappa}{m}$, and hence the period as stated.

   If the masses were hanging vertically, the same equation will apply to oscillations about the equilibrium position. At the equilibrium, where the second spring is stretched by x_0 we have

   $$mg = kx_0,$$

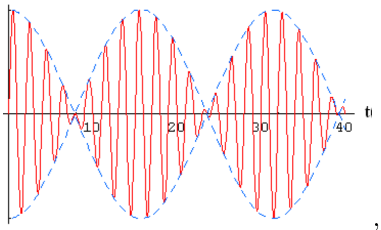
   and the second spring must be stretched by $X_0 = \frac{k}{K}x_0$.

   Note that the presence of gravity would modify the equations of motion as

   $$m(\ddot{x} + \ddot{X}) = -kx + mg, \qquad kx = KX,$$

   which means that the quantities $x - x_0$, $X - X_0$ would satisfy the original system of equations, without mg.

3. Show that the solution of
$$\ddot{x} + \omega^2 x = f \cos{(\omega + \epsilon)t}$$
with $x = \dot{x} = 0$ can be written as
$$x = \frac{2f}{\epsilon^2 + 2\omega\epsilon} \sin\frac{1}{2}\epsilon t \, \sin(\omega + \frac{1}{2}\epsilon)t.$$
(Use the formula $\cos u - \cos v = -2\sin\frac{u+v}{2}\sin\frac{u-v}{2}$.)
Sketch qualitatively this solution for a small ϵ.
Discuss what happens to it when ϵ goes to zero when resonance is achieved.
Answer: The particular solution corresponding to to the forcing $fe^{i\Omega t}$ is found by trying $x_p(t) = Ke^{i\Omega t}$ for some complex constant K, only to discover that K is real, and
$$x_p(t) = \frac{f_0}{\omega^2 - \Omega^2} e^{i\Omega t}.$$
So, the response to the cosine forcing $f \cos \Omega t$ is obtained by taking the real part of the above, which is $x_p(t) = \frac{f_0}{\omega^2 - \Omega^2} \cos \Omega t$.
This has to be now combined with the solution of the homogeneous solution
$$x_h(t) = A\cos\omega t + B\sin\omega t$$
to find the constants. From zero initial conditions:
$$\frac{f_0}{\omega^2 - \Omega^2} + A = 0, \qquad B = 0.$$
Hence
$$x(t) = x_p(t) + x_h(t) = \frac{f_0}{\omega^2 - \Omega^2}(\cos\Omega t - \cos\omega t),$$
and the answer now follows from
$$\cos u - \cos v = -2\sin\frac{u+v}{2}\sin\frac{u-v}{2}.$$
See http://oakroadsystems.com/twt/sumdiff.htm for many helpful trig formulae.
The graph is the beats



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incorporating a slow "modulation" with frequency $\epsilon/2$ and fast "carrier frequency" $\omega + \frac{1}{2}\epsilon \approx \omega$. As we're dealing with the product of the sines, the vertical axis in the figure should actually pass through one of the nodes.
When ϵ goes to zero, for any given t, $\sin\frac{1}{2}\epsilon t$ goes to $\frac{1}{2}\epsilon t$. Using this and omitting ϵ^2 in the denominator in
$$x = \frac{2f}{\epsilon^2 + 2\omega\epsilon} \sin\frac{1}{2}\epsilon t \, \sin(\omega + \frac{1}{2}\epsilon)t,$$
as well as omitting the term $\frac{1}{2}\epsilon$ under the last sine, we get
$$x(t) = \frac{f}{2\omega} t \sin(\omega t),$$
the resonance solution.

4. A damped harmonic oscillator with constants γ, ω is forced by a sine force with the frequency Ω. Find the value of Ω where the steady state solution has maximum amplitude.

Suppose, at the time $t = 1000$, after forcing has been on for a long time, the solution $x(t)$ passes through a local maximum. At this very moment the external forcing gets suddenly switched off. Sketch $x(t)$ qualitatively before and after this has happened.

The steady state solution in response to $fe^{i\Omega t}$ is (after trying the solution $x_p(t) = Ke^{i\Omega t}$ for some *complex* K, ie with $K = Ae^{i\psi}$, with A and ψ to be determined)

$$x_p(t) = \frac{f_0}{\sqrt{(\omega^2 - \Omega^2)^2 + 4\Omega^2\gamma^2}} e^{i\left(\Omega t - \arctan\frac{2\gamma\Omega}{\omega^2 - \Omega^2}\right)}$$

If the forcing is sinusoidal, we take the imaginary part of the above, i.e. the the complex exponential above gets replaced by the sine, to get the steady state solution.

Let us now look at the amplitude only, i.e the pre-exponential factor in $x_p(t)$. Differentiating the square of the denominator

$$(\omega^2 - \Omega^2)^2 + 4\Omega^2\gamma^2$$

with respect to Ω^2 yields a single critical point at

$$\Omega^2 = \omega^2 - 2\gamma^2.$$

Thus, it must be a maximum of the amplitude, because the amplitude is positive and vanishes as $\Omega \to \infty$.

At the time $t = 1000$ forcing has been on for a long time,so as $\gamma > 0$ the solution of the homogeneous equation $x_h(t) \sim e^{-\gamma t}$ can be regarded as zero. In other words, before $t = 1000$, we have $x(t) = x_p(t)$, the steady state solution, which is

$$x_p(t) = a\cos(\Omega t - \psi),$$

where a is the amplitude and ψ is irrelevant, as long as we know that ψ is such that for $t = 1000$, $x_p(t) = a$.

After the forcing has been switched off, for $t \geq 1000$, the solution $x(t)$ should satisfy the homogeneous equation. In other words,

$$x(t + 1000) = e^{-\gamma t}(A\cos\tilde{\omega}t + B\cos\tilde{\omega}t.)$$

with, $x(1000) = a$, $\dot{x}(1000) = 0$ (the net solution $x(t)$ and its derivative should be continuous functions). Then $A = a$, $B = a\gamma/\tilde{\omega}$, which can be rewritten as

$$x(t + 1000) = a\frac{\omega}{\tilde{\omega}}e^{-\gamma t}\cos(\tilde{\omega}t - \phi),$$

for some phase ϕ (see Problem 3) which we will further ignore to make mattes simpler.

So, the net solution

$$x(t) = \begin{cases} a\cos(\Omega t - \psi), & t \leq 1000 \\ a\frac{\omega}{\tilde{\omega}}e^{-\gamma(t-1000)}\cos[\tilde{\omega}(t - 1000) - \phi], & t \geq 1000. \end{cases}$$

The sketch is: oscillations with constant amplitude a and frequency Ω before $t = 1000$, passing smoothly into exponentially vanishing oscillations that begin at $x(1000) = a$, and then vanish at exponential rate γ, and frequency $\tilde{\omega} = \sqrt{\omega^2 - \gamma^2}$.