Some applications of Cauchy-Schwarz inequality

The Cauchy-Schwarz inequality states that

$$\sum_{i=1}^{N} a_i b_i \leq \sqrt{\sum_{i=1}^{N} a_i^2} \cdot \sqrt{\sum_{i=1}^{N} b_i^2}. \tag{1}$$

It is often used in the particular case when every $b_i = 1$, then, after squaring, and calling the summation variable as x and a_i as $a(x)$ it becomes

$$\left(\sum_{x=1}^{N} a(x)\right)^2 \leq N \sum_{x=1}^{N} a(x)^2. \tag{2}$$

The Cauchy-Schwarz inequality has many rather spectacular applications for some combinatorial estimates. Here are some.

Set intersections. Let N be a large number. For a finite set S, let $|S|$ denote the cardinality, i.e. the number of elements in S. Let N be a large integer. Let $\delta \in (0,1)$ be small. Namely, δ being small means that we will still regard $1-4\delta$ as being reasonably close to 1, and N being large that N^2 is much bigger than N. We shall also regard N to any power between 0 and 1 as integer. Otherwise, we would have to take integer part of such numbers, which would only necessitate more notations, without violating the estimates that follow.

Suppose, there are $N^{1-\delta}$ distinct subsets S_i of S, such that every $|S_i| = N^{1-\delta}$. Note that $\frac{|S_i|}{|S|} = N^{-\delta}$, which is still a small number, so each S_i alone is only a very small fraction of S. But there are many of them. So, let us show that

$$\exists \text{ non-equal } i, j: \quad |S_i \cap S_j| \geq \frac{1}{2} N^{1-2\delta}. \tag{3}$$

I.e., the intersection $S_i \cap S_j$ is also quite big in size; as for the constant $\frac{1}{2}$ multiplying the "important term" in the right-hand side of the estimate, it can be anything smaller than 1 and going to 1, for N large enough. In the sequel, i, j always run from 1 to $N^{1-\delta}$ and the variable x runs over S, without putting this explicitly.

Let us introduce characteristic functions $f_i(x)$ of sets S_i as follows: for any x,

$$f_i(x) = \begin{cases} 1 \text{ if } x \in S_i, \\ 0 \text{ otherwise.} \end{cases} \tag{4}$$

Note some of their properties:

$$f_i(x) = f_i^2(x), \qquad \sum_x f_i(x) = |S_i|, \qquad \sum_x f_i(x) f_j(x) = |S_i \cap S_j|. \tag{5}$$

In particular,

$$\sum_i \sum_x f_i(x) = \sum_i |S_i| = N^{2-2\delta}.$$

Hence,

$$N^{2-2\delta} = \sum_x \left(\sum_i f_i(x)\right).$$

Now apply (2) to the summation in x above, with $a(x)$ being the expression in brackets:

$$N^{4-4\delta} \leq N \sum_x \left(\sum_i f_i(x)\right)^2 = N \sum_x \left(\sum_{i,j} f_i(x) f_j(x)\right) = N \sum_{i,j} \left(\sum_x f_i(x) f_j(x)\right) = N \sum_{i,j} |S_i \cap S_j|.$$

There are two options in the double sum: $i = j$ and $i \neq j$, and the number of terms with $i \neq j$ is much bigger than with $i = j$. If $i = j$, $\sum_{i,j} |S_i \cap S_j| = \sum_i |S_i| = N^{2-2\delta}$. This times N is much less than the left-hand side $N^{4-4\delta}$. So, as N is large, we can continue, with any constant $C>1$ in the right-hand side, as

$$N^{3-4\delta} \le C\sum_{i\neq j} |S_i\cap S_j|. \tag{6}$$

Now, we use the "pigeonhole principle". If 13 pigeons are to sit on 12 pigeonholes, there must be a pigeonhole with more than one pigeon sitting on it. In other words, there is a pigeonhole with *at least* the average number of pigeons on it. Apply this principle to (6). For different $i\neq j$, we have the sum of $N^{2-2\delta}-N^{\delta}\ge cN^{2-2\delta}$ terms, where c can be any constant <1 and going to 1 for large N. This sum is $\ge \frac{1}{C}N^{3-4\delta}$, for any $C>1$. So, there must be a term, which has at least the average magnitude, that is for some (i,j):

$$|S_i\cap S_j| \ge \frac{cN^{3-4\delta}}{CN^{2-2\delta}} \ge \frac{1}{2}N^{1-2\delta},$$

because C can be as close to 1 from above as we please, and c can be as close to 1 as we please from below.

Point-line incidence bound. Suppose we have a large number N of points, as well as N straight lines in the plane. In the sequel, let's call the set of points P and the set of lines L. Lowercase p,l will denote individual members of these sets, respectively. The aim is to get is a reasonable upper bound for the number of incidences I between lines in L and points in P, defined as

$$I=\sum_{p,l}\delta_{pl} \qquad \text{with} \qquad \delta_{pl}=\begin{cases} 1 \text{ if } p\in l, \\ 0 \text{ otherwise.}\end{cases}$$

In other words, if $n(p)$ is the number of lines from L passing through a point $p\in P$, or $n(l)$ is the number of points of P supported on the line $l\in L$, then $I=\sum_p n(p)=\sum_l n(l)$.

A straightforward estimate, thinking that every point belongs to every line is

$$I\le N^2. \tag{7}$$

But we can do better than that. Apply Cauchy-Schwartz (2) as follows:

$$I^2=\left(\sum_p\left(\sum_l \delta_{pl}\right)\right)^2 \le N\sum_p\left(\sum_l\delta_{pl}\right)^2 = N\sum_p\sum_{l,l'}\delta_{pl}\delta_{pl'} = N\sum_{l,l'}\left(\sum_p \delta_{pl}\delta_{pl'}\right). \tag{8}$$

The sum over l,l' is the sum over all ordered pairs (l,l') of lines. Given a pair (l,l'), the quantity $\sum_p \delta_{pl}\delta_{pl'}$ is the number of points of P which lie simultaneously on l and on l'.

There are again two cases: $l=l'$ and $l\neq l'$. If $l=l'$, then

$$\sum_{l=l'}\left(\sum_p\delta_{pl}\delta_{pl'}\right)=\sum_l\left(\sum_p\delta_{pl}\right)=I.$$

Otherwise, given a pair $l\neq l'$, the maximum number of points of P lying on both l and l' is 1, because any two distinct lines intersect at no more than one point. Thus (8) becomes

$$I^2\le NI+N\sum_{l\neq l'}1\le 2N^3,$$

because of (7) and the fact that the number of pairs (l,l'), $l\neq l'$ is certainly bounded by N^2. So, we have.

$$I\le\sqrt{2}N^{\frac{3}{2}},$$

which is much better than (7), and it's easy to show that the constant $\sqrt{2}$ can be replaced by any $C>1$.

Fat elephant inequality. Consider a set S of N points in $\mathbb{R}^3$ and look at the projections of S on the coordinate planes xy (the projection going along the z-axis), yz (along the x-axis), and zx (along the y-axis). Let us show that at least one of the projections is such that its size is not less than $N^{2/3}$. (A fat elephant cannot look thin from all the three directions – it must have at last one fat projection.)

Introduce the characteristic function $f(x,y,z)$ of the set S, which equals 1 if the point $(x,y,z) \in S$ and $f(x,y,z) = 0$ otherwise. In the same fashion, let $f_1(x,y)$, $f_2(y,z)$, $f_3(z,x)$ be characteristic functions of the projections of the set S onto the xy, yz, zx-planes, respectively. We will use the fact that characteristic functions squared still equal themselves.

Then the starting point is the claim

$$f(x,y,z) \leq f_1(x,y) f_2(y,z) f_3(z,x). \tag{9}$$

This merely says: a member of S *has* its projections. Namely, $f(x,y,z) = 1$ only if each $f_1(x,y), f_2(y,z), f_3(z,x)$ equals 1 (it is not necessarily true the other way around). Besides,

$$\sum_{x,y,z} f(x,y,z) = N. \tag{10}$$

Here x belongs to the finite set of abscissae of the points of S, y is in the finite set of ordinates of these points, and so on, but we will never have to deal with these sets explicitly.

Let us use (9, 10) and Cauchy-Scwartz (1) applied twice:

First, we apply (1) to summation in (x,y):

$$N \leq \sum_{x,y} f_1(x,y) \left(\sum_z f_2(y,z) f_3(z,x) \right) \leq \left(\sum_{x,y} f_1^2(x,y) \right)^{1/2} \cdot \left(\sum_{x,y} \left(\sum_z f_2(y,z) f_3(z,x) \right)^2 \right)^{1/2}.$$

In the first multiplier,

$$\sum_{x,y} f_1^2(x,y) = \sum_{x,y} f_1(x,y) = |P_{xy}(S)|,$$

where $|P_{xy}(S)|$ denotes the size of the projection of S onto the xy-plane.

In the second multiplier, given (x,y) apply (1) to the summation in z:

$$\left(\sum_z f_2(y,z) f_3(z,x) \right)^2 \leq \sum_z f_2^2(y,z) \cdot \sum_z f_3^2(z,x) = \sum_z f_2(y,z) \cdot \sum_z f_3(z,x)$$

So, we have

$$\sum_{x,y} \left(\sum_z f_2(y,z) f_3(z,x) \right)^2 \leq \sum_{x,y} \sum_z f_2(y,z) \cdot \sum_z f_3(z,x) = \sum_{y,z} f_2(y,z) \cdot \sum_{x,z} f_3(z,x) = |P_{yz}(S)||P_{xz}(S)|,$$

where $|P_{yz}(S)|$, $|P_{xz}(S)|$ denote the size of the projection of S onto the yz and xz-planes respectively. Thus, altogether

$$N^2 \leq |P_{xy}(S)||P_{yz}(S)||P_{xz}(S)|,$$

the product of the sizes of the three projections, hence one of them must be is greater than $N^{2/3}$.

Note, the inequality is sharp, take S as the "lattice cube" $[1,\ldots,M] \times [1,\ldots,M] \times [1,\ldots,M]$. The size of each projection is M^2, while S itself has size M^3.