MODELLING AND ANALYSIS OF AD-HOC NETWORKS

Part III: From infinite to finite networks

August 25, 2015

Carl P. Dettmann

with Justin Coon and Orestis Georgiou

Outline

1. *Asymptotic results and how to make sense of them*

2. *Boundary effects: when and how important are they?*

3. *Basic boundary components in two and three dimensions*

4. *Universality and general formulas*

5. *Example for house domain (using MIMO model)*

6. *Complex and fractal geometries*

Full connectivity: Corners, edges and faces, JC, CPD and OG, J Stat Phys 2012

Connectivity in dense networks confined within right prisms, JC, OG and CPD, SpaSWiN 2014

Connectivity of networks with general connection functions, CPD and OG, arxiv:1411.3617

1. Asymptotic results

System model: *Wireless devices (nodes) Poisson distributed with density* ρ *within domain* $\mathcal{V} = L\mathcal{V}_1 \subset \mathbb{R}^d$ *having volume* $V = L^d V_1$*. Pairwise connections independent and have probability* $H(r_{ij})$*. Beamforming and interference are neglected for now.*

There are several rigorous results on connectivity using large network limits for example due to Penrose (1997) and Gupta & Kumar (1999) for the unit disk connection function (surveyed in Walters 2011), Mao & Anderson (2011-14), Penrose (2015) for more general connection functions.

Full connection probability: *For example, Penrose has, for* $\rho \to \infty$ *and* $L \to \infty$ *so that the limit exists, and* $d \geq 2$*,*

$$P_{fc} \to \exp\left(-\rho \int_{\mathcal{V}} \exp\left[-\rho \int_{\mathcal{V}} H(r_{12}) d\mathbf{r}_2\right] d\mathbf{r}_1\right)$$

with strong restrictions on $H(r)$*.*

Example: *Unit disk range* r_0*,* $d = 2$*, flat torus of side length* L*. We find*

$$P_{fc} \to \exp\left(-\rho L^2 \exp\left[-\rho \pi r_0^2\right]\right)$$

and in particular, for convergence, L *must grow exponentially with* ρ*.*

Some remarks

Isolated nodes *The formula for full connectivity results from two main ideas:*

1. *Connectivity is controlled by isolated nodes. Proved for a very restricted class of connection functions (eg requiring compact support), but probably true more generally. Not true for $d = 1$.*

2. *When $\rho \to \infty$ isolated nodes are rare, almost independent, and almost Poisson distributed. Proved for many connection functions of interest.*

Geometries *Very few geometries are considered in the rigorous literature, mostly the flat torus (no boundaries) and d-cube. Mao and Anderson (2012) point out that for very long range $H(r) \approx (r \log r)^{-2}$ the nodes can sense the full domain; for exponentially decaying $H(r)$ this can be ignored.*

Other network features *As well as connectivity, features such as k-connectivity, percolation, coverage have been considered.*

Alternative scalings

Alternative scaling limits are considered in the literature, which allow the connection range to vary, ie

$$H(r) = g(r/r_0)$$

with fixed function $g : \mathbb{R}^+ \cup \{0\} \to [0,1]$. *Examples (Mao & Anderson, 2012):*

Dense network model *Fix* L, *and take* $\rho \to \infty$ *and* $r_0 \approx (\log \rho/\rho)^{1/d} \to 0$.

Extended network model *Fix* ρ, *and take* $L \to \infty$ *and* $r_0 \approx (\log L)^{1/d} \to \infty$.

If all quantities are scaled consistently, the results are equivalent. So, given $\mu \in \mathbb{R}^+$, *a random geometric graph* $\mathcal{G}(\rho, L, r_0)$ *is connected with the same probability as* $\mathcal{G}(\mu^{-d}\rho, \mu L, \mu r_0)$.

Infinite to finite

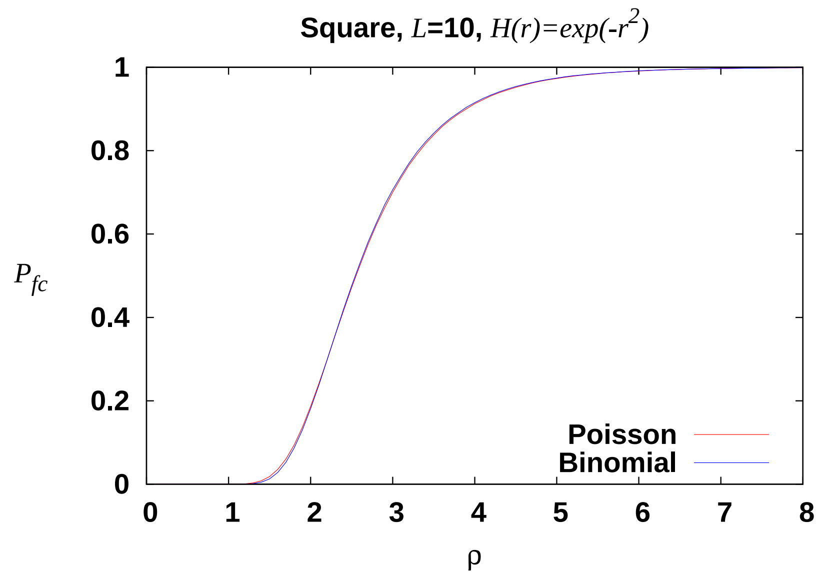
We derive and use the formula

$$P_{fc} \approx \exp\left(-\rho \int_{\mathcal{V}} \exp\left[-\rho \int_{\mathcal{V}} H(r_{12}) d\mathbf{r}_2\right] d\mathbf{r}_1\right)$$

as an approximation, not a limit - good for large but finite ρ *and* L.

Binomial vs Poisson

$Bin(N)$: N *nodes uniform in* $\mathcal{V}$
$Poi(\rho)$: *Density* ρ *in* $\mathcal{V}$.
The latter is equivalent to choosing N *from a Poisson distribution with mean* $\bar{N} = \rho V$ *and then the location of these nodes uniformly. So, we have*

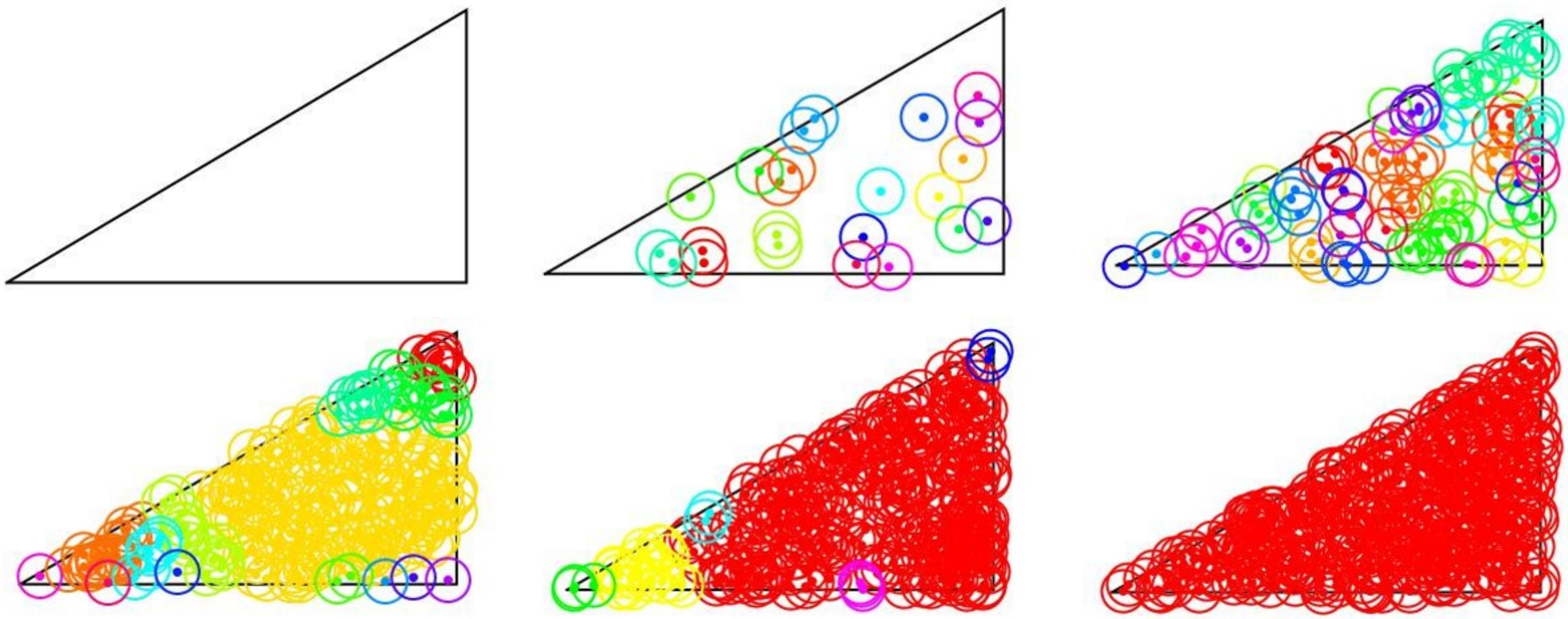


$$P_{fc}^{Poi}(\rho) = \sum_{N=0}^{\infty} \frac{(\rho V)^N e^{-\rho V}}{N!} P_{fc}^{Bin}(N) \approx P_{fc}^{Bin}(\bar{N})$$

2. Boundary effects: When and how important are they?

Asymptotic results *In the above limit ($\rho \to \infty$, $L \to \infty$ so that P_{fc} fixed), the system size grows so fast that isolated nodes are normally in the bulk. So, in theory, boundaries don't matter much.*

But what about increasing density and a fixed (or slowly growing) size?



Isolated nodes occur mostly near the corners!

Boundaries - intuition

Let's look at the connectivity again

$$P_{fc} \approx \exp\left(-\rho \int_{\mathcal{V}} \exp\left[-\rho \int_{\mathcal{V}} H(r_{12}) d\mathbf{r}_2\right] d\mathbf{r}_1\right)$$

If the system size is not growling exponentially fast, the dominant contributions to the outer integral are from minima of the **connectivity mass**

$$M(\mathbf{r}_1) = \int_{\mathcal{V}} H(r_{12}) d\mathbf{r}_2$$

If $\mathbf{r}_1$ *is a point on a boundary* B *with (solid) angle* ω_B*, this separates into angular and radial components:*

$$M(\mathbf{r}_1) \approx M_B = \omega_B H_{d-1}$$

where

$$H_s = \int_0^\infty r^s H(r) dr$$

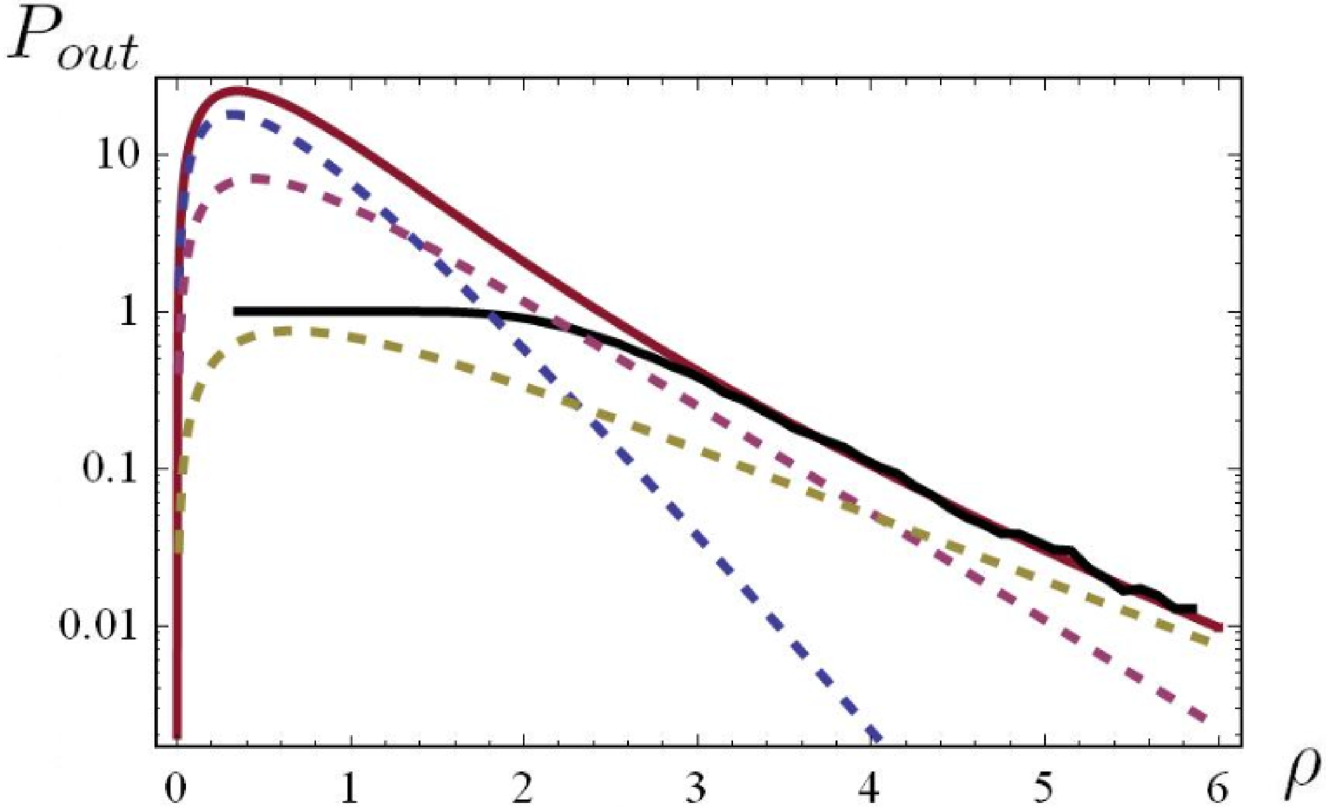
is a moment of the connection function.

Insight: Since the system is much larger than the connection range, we can treat the various boundary components separately and construct many different geometries.

Example: A square

At large ρ we expect the dominant contribution to come from the corners; at smaller ρ a trade-off between the contribution and size of each boundary component. A calculation (to be explained in detail) gives

$$1 - P_{fc} \approx L^2 \rho \exp(-\pi\rho) + \frac{4L}{\sqrt{\pi}} \exp\left(-\frac{\pi\rho}{2}\right) + \frac{16}{\pi\rho} \exp\left(-\frac{\pi\rho}{4}\right)$$



3. Basic boundary components in 2D and 3D

Now we analyse the integrals defining P_{fc} - but it turns out we only need to do this once, as the formulas are quite general. In the following, boundary components are labelled by (d, i), the dimension of the whole space, and the codimension of the boundary component.

Step 1: Integration on a non-centred line

$$F(x) = \int_0^\infty H(\sqrt{x^2 + t^2})dt$$

Expanding in powers of x, taking care with any discontinuities, we find

$$F(x) = H_0 + \frac{x^2}{2}\left(H'_{-1} + \Delta_{-1}\right) + \ldots$$

where H_0 is the zeroth moment, and

$$H'_{-1} = \int_0^\infty \frac{H'(r)}{r}dr = H_{-2}$$

using integration by parts, if the latter converges.

$$\Delta_{-1} = \sum_k \frac{H(r_k+) - H(r_k-)}{r_k}$$

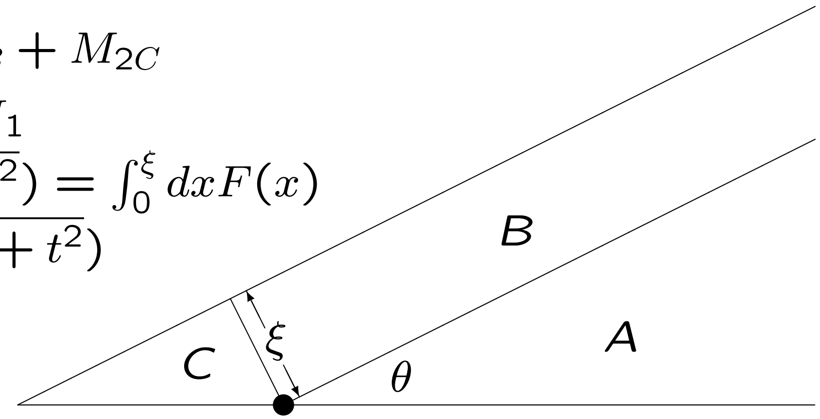
where the sum is over discontinuities (as in the unit disk model). It is convenient to combine these in the notation to write

$$\tilde{H}_{-2} = H'_{-1} + \Delta_{-1}$$

Step 2: Connectivity mass of a wedge

Define $M^{\omega}_{2,2}(r,\theta)$ *to be connectivity mass of a wedge of angle* ω *from a point at polar coordinates* (r,θ)*.*

$$M^{\theta}_{2,2}(\xi \csc\theta, 0) = M_{2A} + M_{2B} + M_{2C}$$
$$M_{2A} = \int_0^{\theta} d\phi \int_0^{\infty} H(r)dr = \theta H_1$$
$$M_{2B} = \int_0^{\xi} dx \int_0^{\infty} dt H(\sqrt{x^2+t^2}) = \int_0^{\xi} dx F(x)$$
$$M_{2C} = \int_0^{\xi} dx \int_0^{x\cot\theta} dt H(\sqrt{x^2+t^2})$$
$$\approx \tfrac{1}{2} H(0)\xi^2 \cot\theta$$



Putting it together we have for this wedge

$$M^{\theta}_{2,2}(\xi\csc\theta, 0) = \theta H_1 + \xi H_0 + \frac{\xi^2}{2} H(0)\cot\theta + \frac{\xi^3}{6}\tilde{H}_{-2} + \dots$$

From this we can find a general wedge, edge and bulk:

$$\begin{aligned}
M^{\omega}_{2,2}(r,\theta) &= M^{\theta}_{2,2}(r,0) + M^{\theta'}_{2,2}(r,0) \qquad (\theta' = \omega - \theta) \\
&= \omega H_1 + r H_0(\sin\theta + \sin\theta') + \frac{r^2}{4} H(0)(\sin 2\theta + \sin 2\theta') \\
&\quad + \frac{r^3}{6}\tilde{H}_{-2}(\sin^3\theta + \sin^3\theta') + \dots \\
M_{2,1}(r) &= 2M^{\pi/2}_{2,2}(r,0) = \pi H_1 + 2rH_0 + \frac{r^3}{3}\tilde{H}_{-2} + \dots \\
M_{2,0} &= 2\pi H_1
\end{aligned}$$

Step 3: Calculation of the outer integral

Here, we use Laplace's method, treating ρ as the large parameter. For example, a wedge of angle ω:

$$
\begin{aligned}
P_{2,2}^{\omega} &= \rho \int_{w} e^{-\rho M_{2,2}^{\omega}(r,\theta)} r dr d\theta \\
&= \rho \int_0^{\omega} d\theta \int_0^{\infty} r dr e^{-\rho\left[\omega H_1 + r H_0(\sin\theta + \sin\theta') + \frac{H(0)r^2}{4}(\sin 2\theta + \sin 2\theta') + \frac{\tilde{H}_{-2} r^3}{6}(\sin^3\theta + \sin^3\theta') + \ldots\right]} \\
&= \rho e^{-\rho\omega H_1} \int_0^{\omega} d\theta \int_0^{\infty} r dr e^{-\rho r H_0(\sin\theta + \sin\theta')} \\
&\quad \left[1 - \frac{\rho H(0) r^2}{4}(\sin 2\theta + \sin 2\theta') - \frac{\rho \tilde{H}_{-2} r^3}{6}(\sin^3\theta + \sin^3\theta') + \ldots\right] \\
&= e^{-\rho\omega H_1} \int_0^{\omega} d\theta \\
&\quad \left[\frac{1}{\rho H_0^2(\sin\theta + \sin\theta')^2} - \frac{3H(0)(\sin 2\theta + \sin 2\theta')}{2\rho^2 H_0^4(\sin\theta + \sin\theta')^4} - \frac{4\tilde{H}_{-2}(\sin^3\theta + \sin^3\theta')}{\rho^3 H_0^5(\sin\theta + \sin\theta')^5} + \ldots\right] \\
&= e^{-\rho\omega H_1} \left[\frac{1}{\rho H_0^2 \sin\omega} - \frac{H(0)(2\cos\omega + 1)}{\rho^2 H_0^4 \sin^2\omega} - \frac{2\tilde{H}_{-2}}{\rho^3 H_0^5 \sin\omega} + \ldots\right]
\end{aligned}
$$

4. Universality and general formulas

We can do similar calculations for 3D boundary components including corners with a right angle, which all have a similar form:

$$\begin{aligned} P_{fc} &\approx \exp\left(-\sum_{i=0}^{d}\sum_{b\in\mathcal{B}_i} P_{d,i}^{(b)}\right) \\ &\approx \exp\left(-\sum_{i=0}^{d}\sum_{b\in\mathcal{B}_i} \rho^{1-i} G_{d,i}^{(b)} V_b \exp\left[-\rho\omega_b H_{d-1}\right]\right) \end{aligned}$$

where V_b is the $d-i$ dimensional volume of the boundary component, and the geometrical factor $G_{d,i}^{(b)}$ is given by

$G_{d,i}^{\omega}$	$i=0$	$i=1$	$i=2$	$i=3$
$d=2$	1	$\frac{1}{2H_0}$	$\frac{1}{H_0^2 \sin\omega}$	
$d=3$	1	$\frac{1}{2\pi H_1}$	$\frac{1}{\pi^2 H_1^2 \sin(\omega/2)}$	$\frac{4}{\pi^2 H_1^3 \omega \sin\omega}$

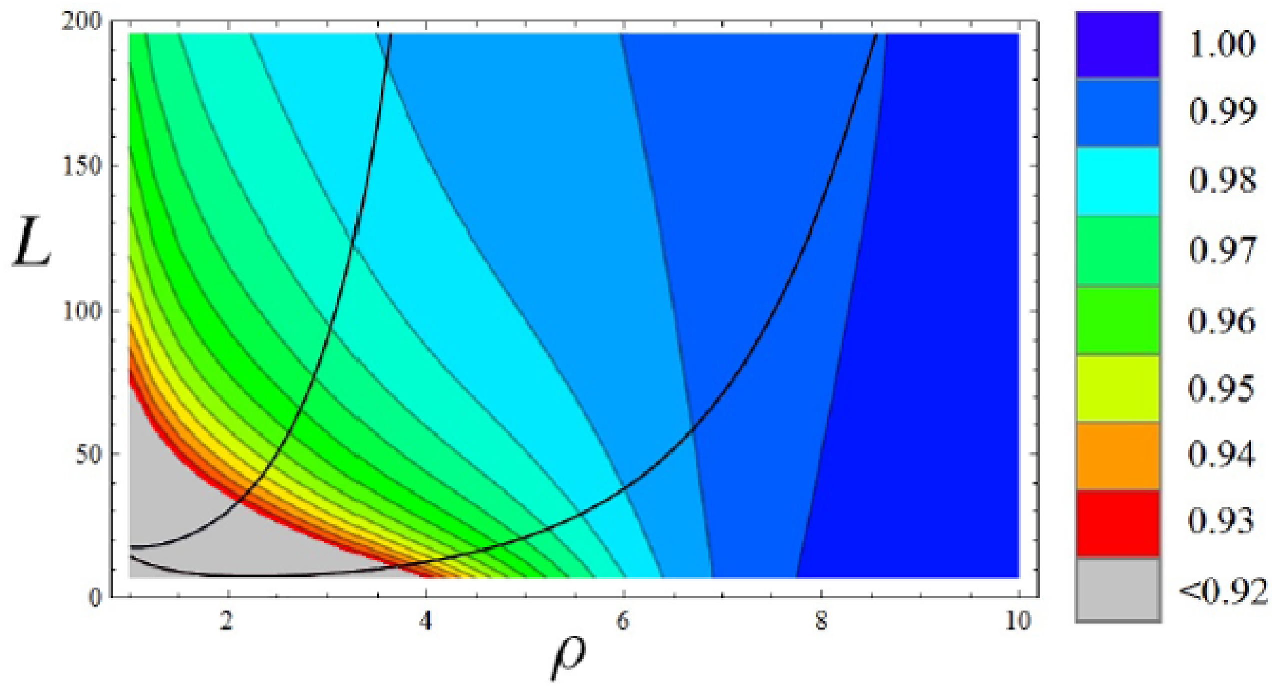
We now have all the ingredients to find P_{fc} for arbitrary convex polygons and right polyhedra.

The square revisited

Recall,

$$1 - P_{fc} \approx L^2 \rho \exp(-\pi\rho) + \frac{4L}{\sqrt{\pi}} \exp\left(-\frac{\pi\rho}{2}\right) + \frac{16}{\pi\rho} \exp\left(-\frac{\pi\rho}{4}\right)$$

We can test convergence as $\rho \to \infty$ *and* $L \to \infty$ *by plotting* $\frac{1-P_{fc}}{\sum_B \cdots}$



Summary so far

Given

- *A connection function* $H(r)$ *corresponding to a specific fading model,*
- *A convex polygonal or polyhedral geometry*

We need to calculate only a few moments

$$H_m = \int_0^\infty r^m H(r) dr$$

and refer to our table of geometrical factors

$G^{\omega}_{d,i}$	$i = 0$	$i = 1$	$i = 2$	$i = 3$
$d = 2$	1	$\frac{1}{2H_0}$	$\frac{1}{H_0^2 \sin\omega}$	
$d = 3$	1	$\frac{1}{2\pi H_1}$	$\frac{1}{\pi^2 H_1^2 \sin(\omega/2)}$	$\frac{4}{\pi^2 H_1^3 \omega \sin\omega}$

to find a good approximation for the full connection probability

$$P_{fc} \approx \exp\left(-\sum_{i=0}^{d} \sum_{b \in \mathcal{B}_i} \rho^{1-i} G^{(b)}_{d,i} V_b \exp\left[-\rho \omega_b H_{d-1}\right] \right)$$

5. Detailed example: "House" with MIMO connection

2×2 MIMO MRC channel with path loss $\eta = 2$:

$$H(r) = e^{-\beta r^2} \left(\beta^2 r^4 + 2 - e^{-\beta r^2} \right)$$

We have for the moments $H_m = \int_0^\infty r^m H(r) dr$:

$$H_2 = \frac{23 - \sqrt{2}}{16} \sqrt{\frac{\pi}{\beta^3}}$$

$$H_1 = \frac{7}{4\beta}$$

Thus the geometrical factors are

Bulk: $G_{3,0} = 1$

Surface: $G_{3,1} = \frac{1}{2\pi H_1} = \frac{2\beta}{7\pi}$

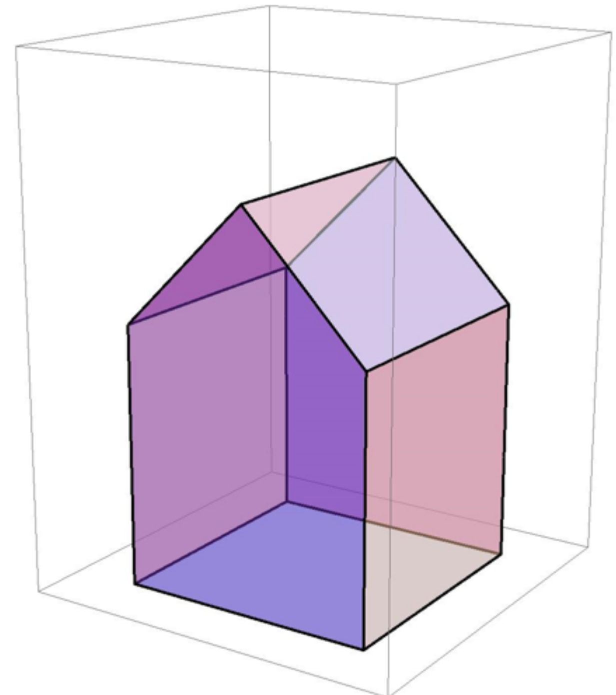
Edge angle θ: $G_{3,2}^{2\theta} = \frac{1}{\pi^2 H_1^2 \sin\theta} = \frac{16\beta^2}{49\pi^2 \sin\theta}$

Corner angle θ: $G_{3,3}^{\theta} = \frac{4}{\pi^2 H_1^3 \theta \sin\theta} = \frac{256\beta^3}{343\pi^2 \theta \sin\theta}$

House geometry

The house is a prism as shown: Base a square of side L, apex a right angle, and the total height $3L/2$. Boundary components are:

- *Bulk,* $V_b = \frac{5}{4}L^3$
- *Surface,* $V_b = \frac{11+2\sqrt{2}}{2}L^2$
- *Edges,* $\theta = \pi/2$, $V_b = (9+2\sqrt{2})L$
- *Edges,* $\theta = 3\pi/4$, $V_b = 2L$
- *Corners,* $\theta = \pi/2$, $V_b = 6$
- *Corners,* $\theta = 3\pi/4$, $V_b = 4$



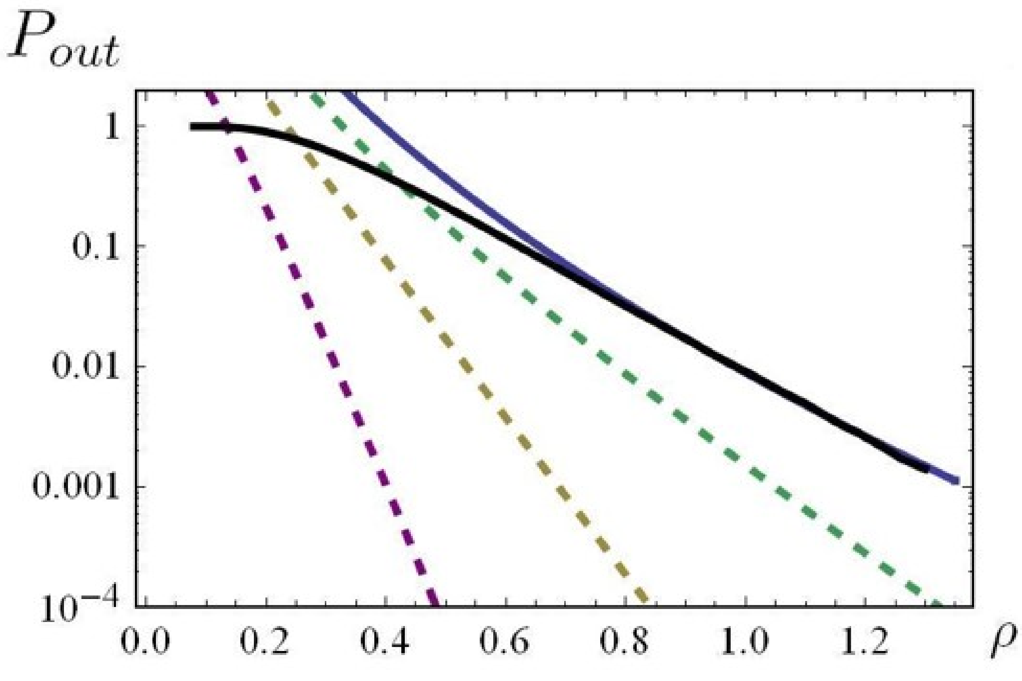
Thus we find $-\ln P_{fc} \approx$

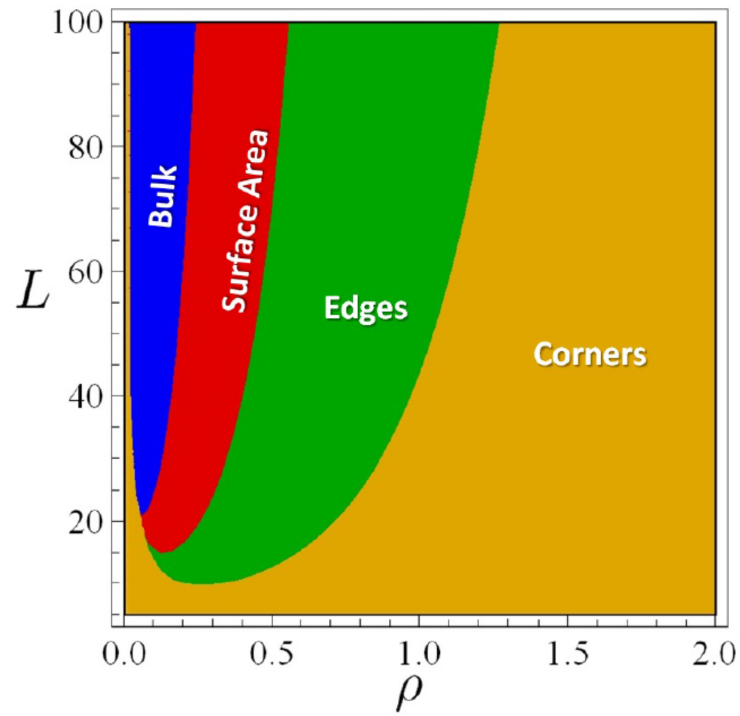
$$\frac{5L^3\rho}{4}\exp\left(-\rho\frac{23-\sqrt{2}}{4}\sqrt{\frac{\pi^3}{\beta^3}}\right)+\frac{(11+2\sqrt{2})\beta L^2}{7\pi}\exp\left(-\rho\frac{23-\sqrt{2}}{8}\sqrt{\frac{\pi^3}{\beta^3}}\right)$$

$$+\frac{16(9+2\sqrt{2})\beta^2 L}{49\pi^2\rho}\exp\left(-\rho\frac{23-\sqrt{2}}{16}\sqrt{\frac{\pi^3}{\beta^3}}\right)+\frac{32\sqrt{2}\beta^2 L}{49\pi^2\rho}\exp\left(-\rho\frac{69-3\sqrt{2}}{32}\sqrt{\frac{\pi^3}{\beta^3}}\right)$$

$$+\frac{3072\beta^3}{343\pi^3\rho^2}\exp\left(-\rho\frac{23-\sqrt{2}}{32}\sqrt{\frac{\pi^3}{\beta^3}}\right)+\frac{4096\sqrt{2}\beta^3}{1029\pi^3\rho^2}\exp\left(-\rho\frac{69-3\sqrt{2}}{64}\sqrt{\frac{\pi^3}{\beta^3}}\right)$$

House: Numerical results



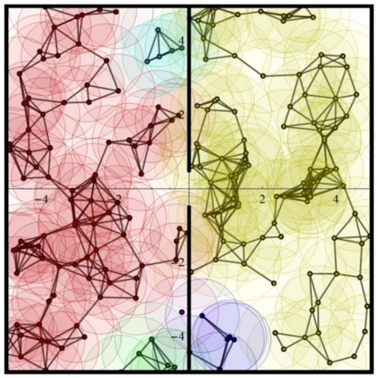


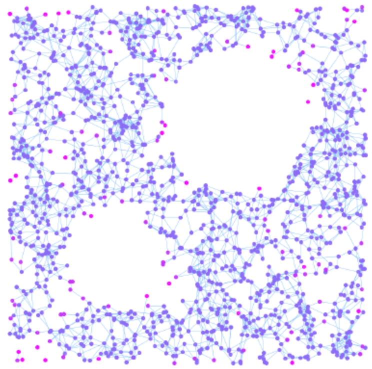
Left: Contributions to the outage probability; direct simulation in black.

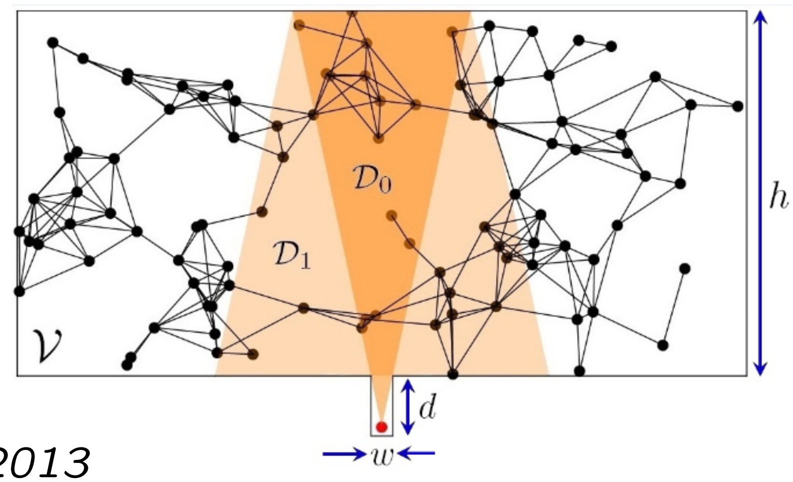
Right: Phase diagram of the dominant contribution.

6. Complex and fractal geometries

These ideas can be extended to non-convex domains...







Keyholes: OG, CPD and JC, ISWCS 2013

Obstacles and curved boundaries: A. P. Giles, OG and CPD, arxiv:1502.05440

Reflections: OG, M. Z. Bocus, M. R. Rahman, CPD, JC,
IEEE Commun Lett 2015

Fractals: CPD, OG and JC, **ISWCS 2015**