Modelling and Analysis of Ad-hoc Networks

Advanced Topics

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General diversity and power scaling laws

Directional antenna network design

Connecting through small openings

Absorption and reflection effects in hallways

General diversity and power scaling laws

Question...

Can we use the observables discussed earlier to investigate how the connectivity properties of a network change with variations in

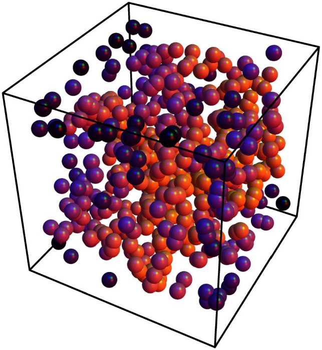
1. the transmit power of each node and
2. the number of antennas employed at each node?

Motivation: Network and transceiver optimisation (e.g., power allocation, MCS scheme).

General diversity and power scaling laws

Network model

- N uniformly distributed nodes (fixed number)
- Node locations: $\mathbf{r}_i \in \mathcal{V} \subseteq \mathbb{R}^d$
- Fixed density $\rho = N/V$ $(V = |\mathcal{V}|)$
- Two nodes i and j a distance $D(\mathbf{r}_i, \mathbf{r}_j)$ apart are directly connected with probability $H(D(\mathbf{r}_i, \mathbf{r}_j))$, which we write as $H(\mathbf{r}_{ij})$ or H_{ij}.



General diversity and power scaling laws

Global vs local

At high node density, recall that

$$P_{fc} \approx 1 - \rho \int_{\mathcal{V}} e^{-\rho \int_{\mathcal{V}} H(\mathbf{r}_{12}) \, \mathrm{d}\mathbf{r}_1} (1 + O(N^{-1})) \, \mathrm{d}\mathbf{r}_2$$

Global connectivity is determined by the *connectivity mass*

$$M[H; \mathbf{r}_2] = \int_{\mathcal{V}} H(\mathbf{r}_{12}) \, \mathrm{d}\mathbf{r}_1$$

Expand M about $\mathbf{r}_2$ situated on a boundary...

$$M \approx \omega \int_0^\infty r^{d-1} H(r) \, \mathrm{d}r$$

where $\omega = \int \mathrm{d}\Omega$ is the solid angle as seen from $\mathbf{r}_2$, with $\Omega = 2\pi^{d/2}/\Gamma(d/2)$ being the full solid angle in d dimensions.

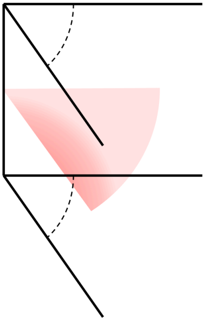
General diversity and power scaling laws

Dependence on solid angle

Local observation...

$$\boxed{M \propto \omega}$$

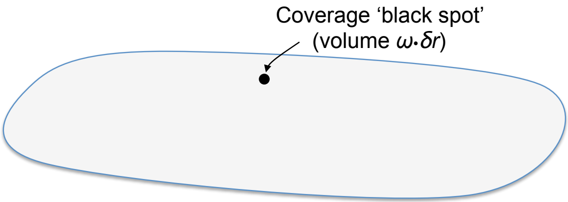
Corresponding global observation...



connectivity depends exponentially on ω.

General diversity and power scaling laws

Dependence on solid angle: analogous view



Assumptions...

- Node situated in black spot cannot connect to nodes outside
- Volume of black spot is $\omega \cdot \delta r$

Probability that no other nodes fall within black spot...

$$\left(1 - \frac{\omega \cdot \delta r}{V}\right)^{N-1} = e^{-\rho\omega\cdot\delta r}(1 + O(N^{-1}))$$

General diversity and power scaling laws

Transmit power P_T

$$P_{fc} \approx 1 - \rho \int_{\mathcal{V}} e^{-\rho M}(1 + O(N^{-1}))\,\mathrm{d}\mathbf{r}_2$$

Approach...

1. Choose a pairwise connection function $H(r)$

$$H(r) = \mathbb{P}(\mathrm{SNR}(r) \cdot X > \mathrm{SNR}_{th}) = 1 - F_X(\beta r^\eta)$$

   where X is the gain of the channel (random), η is the path loss exponent, and $\beta \propto P_T^{-1}$

2. Manipulate the integral $M = \omega \int_0^\infty r^{d-1} H(r)\,\mathrm{d}r$

$$\omega \int_0^\infty r^{d-1}(1 - F_X(\beta r^\eta))\,\mathrm{d}r = \frac{\omega}{\beta^{d/\eta}\eta} \int_0^\infty x^{d/\eta - 1}(1 - F_X(x))\,\mathrm{d}x$$

$$\boxed{M \propto P_T^{d/\eta}}$$

General diversity and power scaling laws

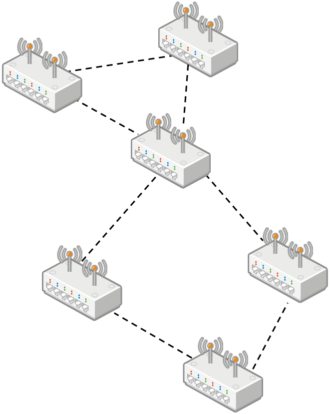
Multi-antenna systems

- Each node has m TX and n RX antennas
- $n \times m$ channel matrix is $\mathbf{H}$, with $h_{ij} \sim \mathcal{CN}(0,1)$
- Two transmission schemes:
  1. STBC

$$\mathrm{SNR}(r) \propto \frac{\zeta_m}{m} \|\mathbf{H}\|_F^2 r^{-\eta}, \quad \zeta_m \in \{1,2\}$$

  2. MIMO-MRC

$$\mathrm{SNR}(r) \propto \lambda_{\max}(\mathbf{H}'\mathbf{H}) r^{-\eta}$$



General diversity and power scaling laws

Multi-antenna systems: STBC

$\|\mathbf{H}\|_F^2$ is chi-squared distributed with $2mn$ degrees of freedom, and thus...

$$M = \frac{\omega}{d}\left(\frac{\zeta_m}{m\beta}\right)^{d/\eta}\frac{\Gamma(mn + d/\eta)}{\Gamma(mn)}$$

which for large m and/or n is

$$M = \frac{\omega}{d}\left(\frac{\zeta_m n}{\beta}\right)^{d/\eta}\left(1 + O\left(\tfrac{1}{mn}\right)\right)$$

So...

$$\boxed{M = O(n^{d/\eta})}$$

For MIMO-MRC, we need the following lemma (Edelman, 1988)...

Lemma

Let $x_n \xrightarrow{p} x$ signify that for all $\varepsilon > 0$, $\lim_{n\to\infty} P(|x - x_n| > \varepsilon) = 0$. Now, suppose the $n \times m$ matrix $\mathbf{H}$ has independent circularly symmetric complex Gaussian entries, each with zero mean and unit variance. Then $\mathbf{W} = \mathbf{H}'\mathbf{H}$ has a complex Wishart distribution and

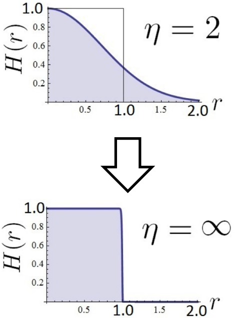
$$(1/n)\lambda_{\max}(\mathbf{W}) \xrightarrow{p} (1+\sqrt{y})^2, \qquad \frac{m}{n} \to y,\ 0 \le y < \infty.$$

General diversity and power scaling laws

Multi-antenna systems: MIMO-MRC

The lemma suggests the approximation

$$H(r) \approx \begin{cases} 1, & r < \left(\frac{(1+\sqrt{y})^2 n}{\beta}\right)^{\frac{1}{\eta}} \\ 0, & \text{otherwise} \end{cases}$$



which leads to

$$M = \frac{\omega}{d}\left(\frac{(1+\sqrt{y})^2 n}{\beta}\right)^{d/\eta}\left(1 + O\left(n^{-\frac{2}{3}}\right)\right), \quad m, n \to \infty, \quad \frac{m}{n} \to y$$

Again...

$$\boxed{M = O(n^{d/\eta})}$$

General diversity and power scaling laws

Multi-antenna systems: STBC vs MIMO-MRC

- For fixed $m = 2$, connectivity is the same for both schemes to leading order.
- Connectivity is dependent on d/η, which gives progressive improvement or diminishing returns.
- If both m and n scale such that $m/n \to y$, $y_c = (\sqrt{2} - 1)^2 \approx 0.172$ denotes a critical ratio for which connectivity is the same for both schemes.
- Different choices of the pairwise connection function lead to different comparisons, but general results (e.g., dependence on d/η) are typically invariant.

General diversity and power scaling laws

Multi-antenna systems: fixed m

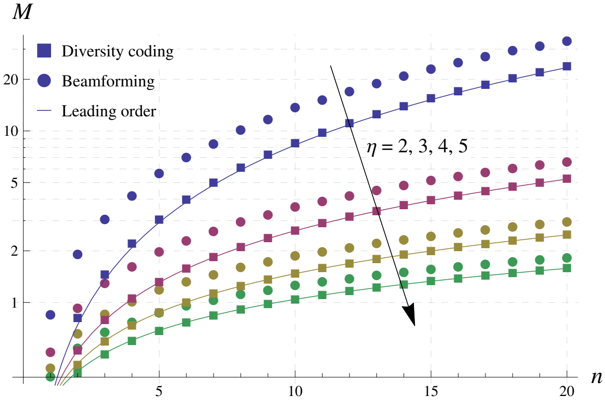


Figure : Connectivity mass vs. n for $m = 2$, $d = 3$, and various values of η, corresponding to connectivity exponents $d/\eta = \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}$. The solid lines correspond to the leading order, whereas the data represented by markers was obtained from exact expressions.

General diversity and power scaling laws

Multi-antenna systems: increasing m

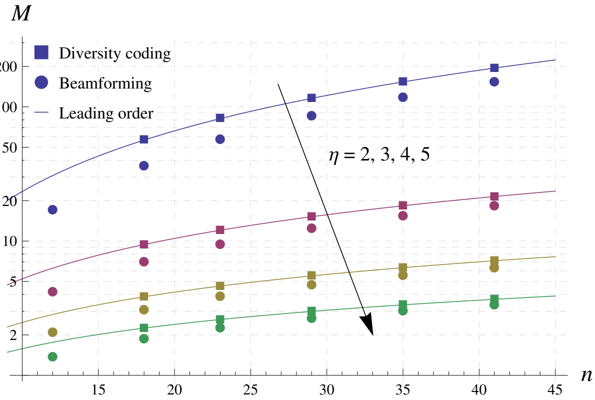


Figure : Connectivity mass vs. n for $m \approx y_c n$, $d = 3$, and various values of η, corresponding to connectivity exponents $d/\eta = \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}$. The solid lines correspond to the leading order, whereas the data represented by markers was obtained from exact expressions.

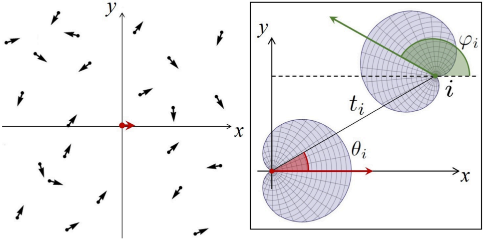
Directional antenna network design

Question...

How does antenna directivity affect the connectivity properties of an *ad hoc* network?

Directional antenna network design

Network model and assumptions

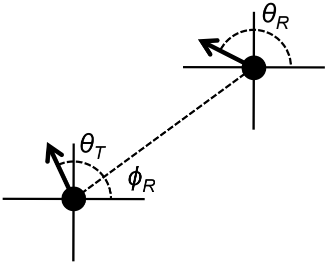


- N nodes in a 2-dimensional space of volume V ($\rho = N/V$)
- Nodes are oriented randomly (uniformly distributed)
- Ignore boundary effects
- SNR between two nodes follows Friis equation

$$\text{SNR} \propto G_T G_R r^{-\eta}$$

Directional antenna network design

Network model and assumptions



- θ_T and θ_R: orientations of nodes
- ϕ_T and $\phi_R = \phi_T + \pi$: observation angles
- TX and RX power is normalised

$$\int_0^{2\pi} G_T(\phi)\,\mathrm{d}\phi = \int_0^{2\pi} G_R(\phi)\,\mathrm{d}\phi = 2\pi$$

Directional antenna network design

Average probability of full connectivity

Pairwise connection function...

$$H(r, \phi_T, \phi_R, \theta_T, \theta_R) = \mathbb{P}(G_T G_R r^{-\eta} X > \text{constant})$$

Probability of full connectivity (to first order)...

$$P_{fc} \approx 1 - Ne^{-\rho M}$$

Anisotropic connectivity mass...

$$M = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\infty} rH(r, \phi, \theta_T, \theta_R) \, \mathrm{d}r \, \mathrm{d}\phi \, \mathrm{d}\theta_R.$$

Directional antenna network design

Anisotropic connectivity mass

Broadly speaking, if the distribution of the channel gain has an exponential tail, then the connectivity mass has the form

$$M = \sum_i a_i \left(\int_0^{2\pi} G_T(\phi)^{\frac{2}{\eta}} \, \mathrm{d}\phi \right) \left(\int_0^{2\pi} G_R(\phi)^{\frac{2}{\eta}} \, \mathrm{d}\phi \right)$$

where $\{a_i\}$ are constants depending on the distribution.

Directional antenna network design

Anisotropic connectivity mass

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1. Effects of directive radiation are somewhat invariant to changes in the underlying channel statistics.

Directional antenna network design

Anisotropic connectivity mass

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where $\{a_i\}$ are constants depending on the distribution.

1. Effects of directive radiation are somewhat invariant to changes in the underlying channel statistics.
2. Connectivity mass is invariant with respect to the gain function for a path loss exponent of $\eta = 2$.

Directional antenna network design

Radiation patterns

Consider three fundamental directivity classes

1. Isotropic
2. Cardioid
3. Beamforming

Let $G_T = G_R = G$ and focus on the functional

$$S_\eta[G] = \int_0^{2\pi} G(\phi)^{\frac{2}{\eta}} \, \mathrm{d}\phi$$

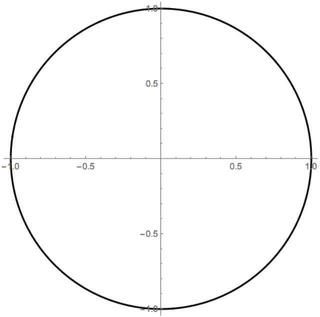
Directional antenna network design

Isotropic radiation

For isotropic radiation, $G(\phi) = 1$ for $0 \leq \phi < 2\pi$, and

$$S_\eta[G] = \int_0^{2\pi} G(\phi)^{\frac{2}{\eta}}\, \mathrm{d}\phi = 2\pi$$

for all η.



Directional antenna network design

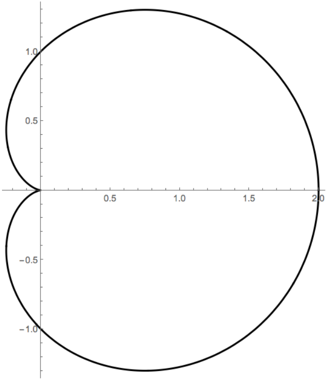
Cardioid radiation pattern

Patch antennas exhibit a cardioid pattern...

$$G(\phi) = 1 + \cos\phi, \quad 0 \le \phi < 2\pi$$

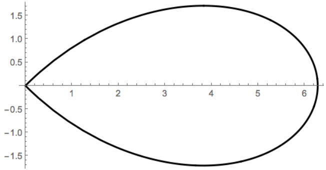
The functional evaluates to...

$$S_\eta[G] = \frac{2^{\frac{2}{\eta}}\eta\sqrt{\pi}\Gamma\left(\frac{1}{2}+\frac{2}{\eta}\right)}{\Gamma\left(\frac{2}{\eta}\right)}$$



Directional antenna network design

Beamforming



A beam forming or end-fire array pattern is captured by the function...

$$G(\phi) = \begin{cases} \lambda\pi \cos(\lambda\phi), & -\frac{\pi}{2\lambda} \leq \phi \leq \frac{\pi}{2\lambda} \\ 0, & \text{otherwise} \end{cases}$$

with $\lambda \geq 1$ indicating the directivity of the beam.

The functional evaluates to...

$$S_\eta[G] = \frac{\pi^{\frac{1}{2}+\frac{2}{\eta}} \lambda^{\frac{2}{\eta}-1} \Gamma\left(\frac{1}{2}+\frac{2}{\eta}\right)}{\Gamma\left(1+\frac{1}{\eta}\right)}$$

Directional antenna network design

$S_\eta[G]$ vs η

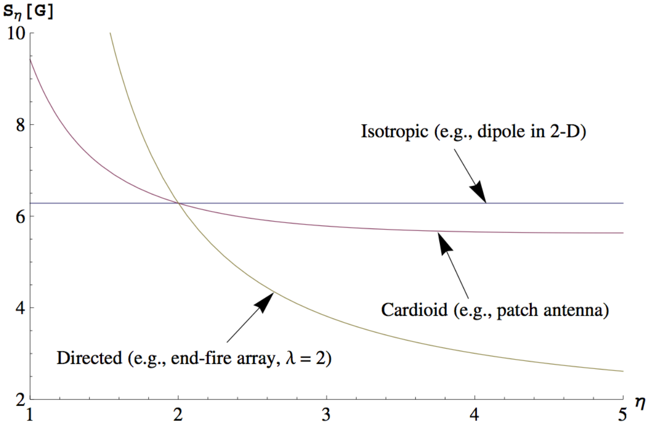
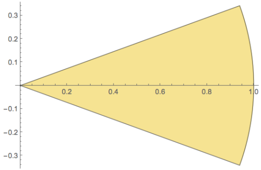


Figure : Illustration of the gain integral $S_\eta[G]$ corresponding to different radiation patterns, plotted as a function of the path loss exponent η.

Directional antenna network design

Gain pattern optimisation: one sector

But what is the *optimal* gain pattern?



1. Let $G(\phi) \geq 0$ on $\phi \in [-\pi/2\lambda, \pi/2\lambda]$ and zero elsewhere
2. Maximise $S_\eta[G]$ subject to the total power constraint

$$\int_0^{2\pi} G(\phi)\,\mathrm{d}\phi = 2\pi$$

3. The stationary path $G^\star$ is given by

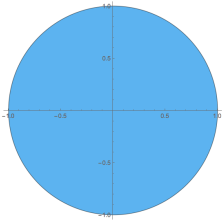
$$G^\star(\phi) = 2\lambda, \quad \phi \in \left[-\frac{\pi}{2\lambda}, \frac{\pi}{2\lambda}\right]$$

4. $G^\star$ maximises connectivity for $\eta > 2$ and minimises connectivity for $\eta < 2$

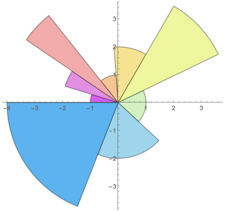
Directional antenna network design

Gain pattern optimisation: conclusions

Corollary: letting $\lambda = 1/2$ yields $G^\star = 1$ and isotropic radiation is optimal for $\eta > 2$.



Extension: analogous results hold when $G(\phi)$ is a nonzero piecewise function on the interval $[0, 2\pi)$

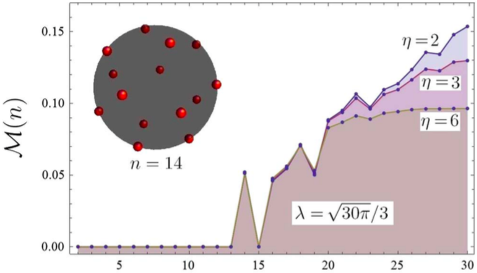


Application: antenna pattern synthesis

Directional antenna network design

Further topics

Effects of anisotropic radiation on the connectivity of **bounded 3D networks** have been thoroughly treated by Georgiou et al (2014).



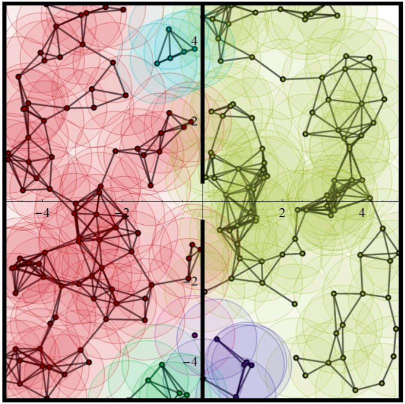
Directivity was shown to improve connectivity in **interference-limited** networks by Georgiou et al (2015).

Connecting through small openings

Problem overview

The problem...

- We have considered convex bounding regions.
- Practical networks (e.g., indoor, urban) are deployed in nonconvex regions.
- Split the domain $\mathcal{V}$ into two regions $\mathcal{A}$ and $\mathcal{B}$ separated by a wall with an opening.



Connecting through small openings

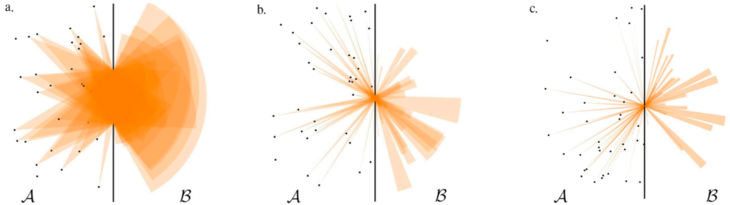
Overview of modelling solution

The approach...

1. Factor connectivity probability into regional connectivity and a bridging link...

$$P_{fc}^{(\mathcal{V})} = P_{fc}^{(\mathcal{A})} P_{fc}^{(\mathcal{B})} X$$

2. Regional connectivity is calculated as shown before.



Connecting through small openings

Overview of modelling solution

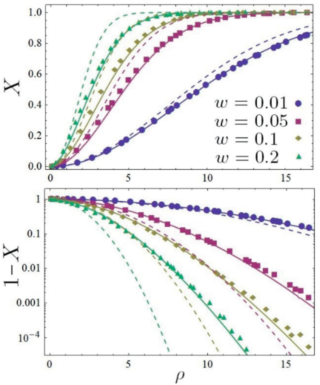
Bridging link is more subtle...

3. Assuming $\mathcal{A} \leftrightarrow \mathcal{B}$ independent...

   $$X \approx 1 - e^{-\rho_{\mathcal{A}}\rho_{\mathcal{B}} \int_{\mathcal{A}} \int_{\mathcal{B}} \chi_{ij} H_{ij} \,\mathrm{d}\mathbf{b}_j \mathrm{d}\mathbf{a}_i}$$

   ...is accurate at low density or for small openings.

4. Conditioning on one region...



$$X = 1 - \left(\frac{1}{V_{\mathcal{A}}} \int_{\mathcal{A}} e^{-\rho_{\mathcal{B}} \int_{\mathcal{B}} \chi_{ij} H_{ij} \,\mathrm{d}\mathbf{b}_j} \,\mathrm{d}\mathbf{a}_i \right)^{N_{\mathcal{A}}}$$

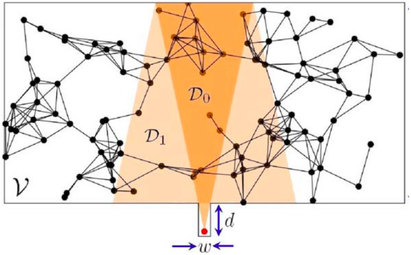
...does not admit a simple form.

Absorption and reflection effects in hallways

Problem overview

The problem...

- We wish to connect "gateway" devices on the exterior of a domain using a "cloud-like" set of devices on the interior.
- Line-of-sight connections can be made.
- Non-line-of-sight connections results from reflections.
- Reflected waves experience step attenuation.



Absorption and reflection effects in hallways

Overview of modelling solution

The approach...

1. Factor connectivity probability into

$$P_{fc}^{\mathcal{V}} P(\text{bridge}) = P_{fc}^{\mathcal{V}} \left(1 - e^{-N\langle H_{ki}\rangle}\right),$$

$$\langle H_{ki}\rangle = \frac{1}{V}\sum_{c=0}^{C}\int_{\mathcal{D}_c} H_{ki}^{(c)}\, \mathrm{d}\mathbf{r}_i$$

2. Reflect the domain and consider different regions.