Sticky and Non-sticky open Mushrooms

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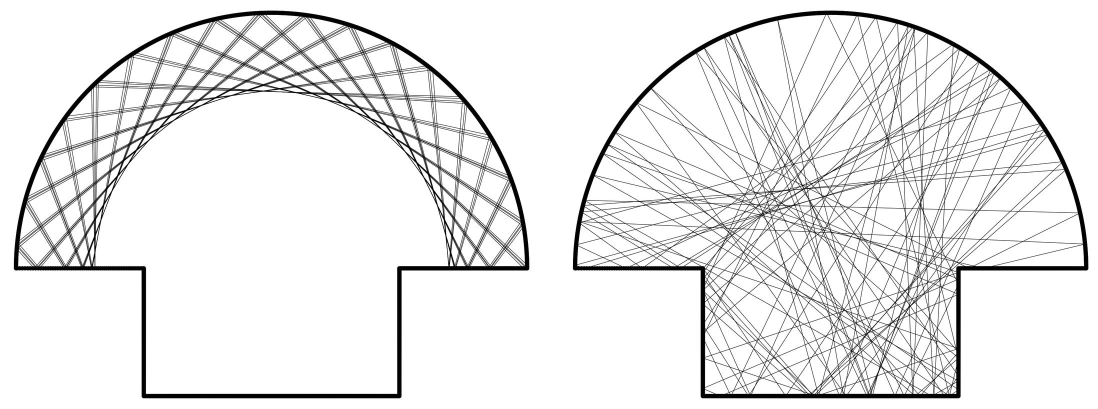
Introduction

A billiard is a dynamical system in which a particle alternates between motion in a straight line and specular reflections from a boundary. The mushroom billiard forms a class of dynamical systems with sharply divided phase in two dimensions. Its mixed phase space is composed of a single completely regular (integrable) component and a single chaotic and ergodic component. For typical values of the control parameter of the system, an infinite number of marginally unstable periodic orbits (MUPOs) exist making the system sticky in the sense that unstable periodic orbits approach regular regions in phase space and thus exhibit regular behaviour for long periods of time. The problem of finding these MUPOs is expressed as the well known problem of finding optimal rational approximations of a number, subject to some system-specific constraints. We introduce a measure zero set of control parameter values for which all MUPOs are destroyed and therefore the system is non-sticky. The open mushroom (billiard with a hole) is considered and the asymptotic survival probability function $P(t)$ is calculated for both cases.

Model

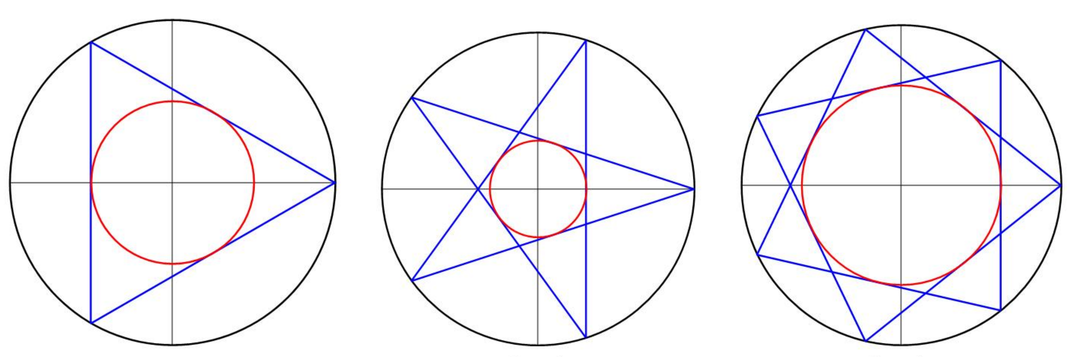


a) Regular b) Chaotic

Stem width: $2r$, Hat radius: R, Control parameter: $\frac{r}{R}$.

What are MUPOs

- The mushroom has infinitely many periodic orbits (POs) living in its hat.
- Each PO forms a star polygon in coordinate space.



$(3,1)$ $(5,2)$ $(7,2)$

- Marginally Unstable Periodic Orbits (MUPOs) are PO which cross the central circle of radius r ⇔ **Sticky Mushroom**.

A perturbed MUPO will begin to precess with a constant precessing angular velocity proportional to the magnitude of the perturbation until the orbit eventually falls into the stem of the mushroom. When this happens, nearby perturbed trajectories will de-correlate by the defocusing mechanism exponentially fast. Until they fall into the stem however, they exhibit intermittent, quasi-regular behaviour causing the system to display weaker statistical properties such as a polynomial decay of correlations and an algebraic asymptotic survival probability.

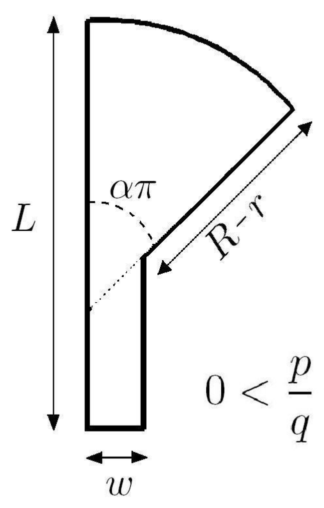
Finding the MUPOs

$$\cos\frac{j\pi}{s} < \frac{r}{R} < \frac{\cos\frac{j\pi}{s}}{\cos\frac{\pi}{\lambda s}}, \qquad (1)$$

where s and j are positive coprime integers such that s is the period of the orbit and j is its rotation number, and λ is 1 if s is even and 2 if odd. Rearranging and expanding for large s:

$$\frac{j}{s} > \vartheta^* > \frac{j}{s} - \left(\frac{\pi\cot\frac{j\pi}{s}}{2}\right)\frac{1}{\lambda^2 s^2} + \mathcal{O}\left(\frac{1}{s^4}\right). \qquad (2)$$

Generalised Mushroom



$$r = \frac{w}{\sin\alpha\pi} \qquad \vartheta^* = \frac{1}{\pi}\arccos\frac{r}{R}$$

$$\alpha \in (0,1]$$

$$0 < \frac{p}{q} - \frac{\vartheta^*}{\alpha} < \frac{\alpha\pi\cot\frac{\alpha p\pi}{q}}{2q^2} < \overbrace{\left(\frac{\alpha\pi r}{2\sqrt{R^2-r^2}}\right)}^{K}\frac{1}{q^2},$$

Note: No λ dependence.
Goal: Want to find values of ϑ^* for which there are no solutions ⇔ **Non-Sticky Mushroom**.

Results

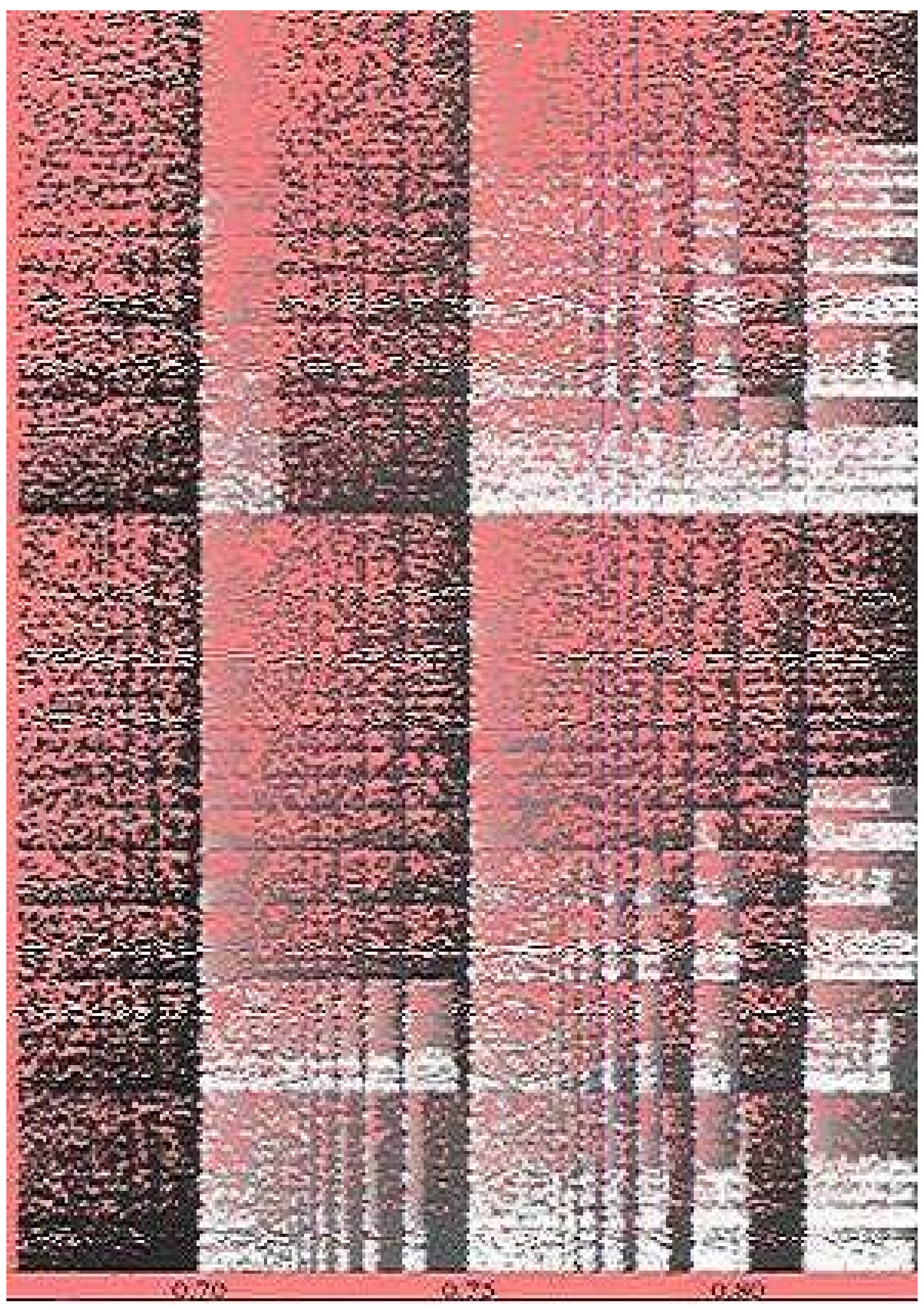
- Use continued fractions representation: $\frac{\vartheta^*}{\alpha} = [a_0; a_1, a_2, \ldots]$.
- If $K \leq \frac{1}{2}$ and $\left|\frac{p}{q} - \frac{\vartheta^*}{\alpha}\right| < \frac{K}{q^2}$, then $\frac{p}{q}$ is always a convergent of $\frac{\vartheta^*}{\alpha}$. Look at convergents $\left(\frac{A_n}{B_n}\right)$ of $\frac{\vartheta^*}{\alpha} = \frac{A_n\zeta_{n+1}+A_{n-1}}{B_n\zeta_{n+1}+B_{n-1}}$, where $\zeta_n = [a_n; a_{n+1}, \ldots]$.
- $\frac{A_n}{B_n} - \frac{\vartheta^*}{\alpha} = \frac{(-1)^{n-1}}{\left(\zeta_{n+1}+\frac{B_{n-1}}{B_n}\right)B_n^2}$. If n is even, then $\frac{A_n}{B_n} - \frac{\vartheta^*}{\alpha} < 0 \Rightarrow$ Look at n odd: $\frac{A_n}{B_n} - \frac{\vartheta^*}{\alpha} > \frac{1}{\left(a_{n+1}+1+\frac{B_{n-1}}{B_n}\right)B_n^2} > \frac{1}{(a_{n+1}+2)B_n^2}$.
- Hence, the mushroom billiard is non-sticky if $K < \frac{1}{C+2}$, where $C = \max a_n$, and n is even.

More Results

Since the set of numbers with bounded partial quotients has **measure zero** ⇔ **Generic Mushrooms are Sticky**. Also, the **Hausdorff dimension** of this set is **one**. We use the following transformation to illustrate this graphically:

$$\vartheta^* \to (x,y) \qquad x = \left[\frac{1}{2}; a_1, \frac{1}{2}, a_3, \ldots\right] \qquad (3)$$

$$y = \left[a_0; \frac{1}{2}, a_2, \frac{1}{2}, \ldots\right] \qquad (4)$$



Darker: Well approximable numbers ϑ^*
Brighter: Badly approximable numbers ϑ^* with respect to eq (2) above.

Survival Probability

Given a density of particles on the billiard boundary, the probability $P(t)$ that a particle survives in a strongly chaotic billiard (*i.e.* does not escape through a small hole of size h) up to time t decays exponentially $\sim e^{-\gamma t}$. However, in the case of the mushroom there is also an integrable island of orbits in phase space which never escapes, a set of orbits called near-bouncing ball orbits which decay algebraically and a set of orbits near the MUPOs (described above) also decaying algebraically. This is summarised bellow:

$$P(t) = \begin{cases} \text{irregular}, & t < t^* \\ e^{-\gamma t} + A_{ii} + \frac{B_{bb}}{t} + \frac{C_{MUPOs}}{t}, & t > t^*. \end{cases} \qquad (5)$$

We have obtained leading order expressions for all the above mentioned constants, In particular:

$$C_{MUPOs} = \sum_{(s,j)\in\mathcal{S}_r} \lambda(s+2j)\Delta_{s,j}, \qquad (6)$$

$$\Delta_{s,j} = \frac{8R\cos^2\theta_{s,j}\left(\pi - s\lambda\arccos\left(\frac{R}{r}\sin\theta_{s,j}\right)\right)^2}{2s^2\lambda^2|\partial Q|t}. \qquad (7)$$

If the mushroom is non-sticky however, $\mathcal{S}_r = \emptyset$ and hence $C_{MUPOs} = 0$.

Conclusion

In conclusion, whether the mushroom billiard is sticky or not affects the overall classical dynamics of the system. Therefore, one may expect a quantum mechanical manifestation of this to be observed in mesoscopic and wave physics. Also, it is interesting in general to consider variable Diophantine conditions (*i.e.* K as a function of ϑ^*)

References

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2. C.P. Dettmann and O. Georgiou, *Sticky and Non-sticky open Mushrooms*, in preparation, (2010).
3. A.M. Rockett and P. Szüsz, *Continued Fractions*, World Scientific, (1992).