Transmission and Reflection in the Stadium Billiard: Time-dependent asymmetric transport.

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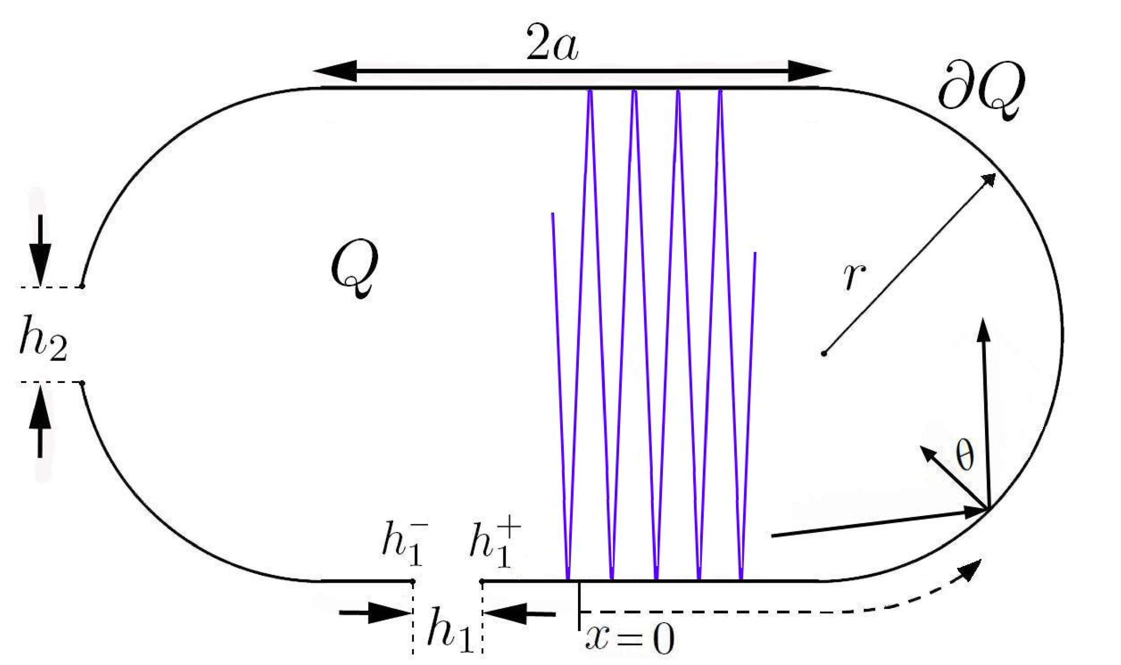
1. Introduction

A billiard is a dynamical system in which a particle alternates between motion in a straight line and specular reflections from a boundary. The stadium billiard is a chaotic system where the defocusing mechanism guarantees a positive Lyapunov exponent λ (exponential separation rate of nearby trajectories) almost everywhere, the exception being a zero-measure family of marginally unstable periodic orbits between the parallel straight segments called Bouncing Ball orbits. They have been shown to lead to an intermittent, quasi-regular behaviour which effectively causes the closed stadium to display some weaker chaotic properties such as an algebraic decay of correlations. Quantum mechanically they cause scarring, the system is not quantum uniquely ergodic, an $\hbar$ dependent 'island of stability' appears to surround them and deviations from random matrix theory (RMT) predictions are observed if not treated appropriately. The question we address here is **what is their effect on Transport through the system?**

2. Model: Stadium with 2 holes

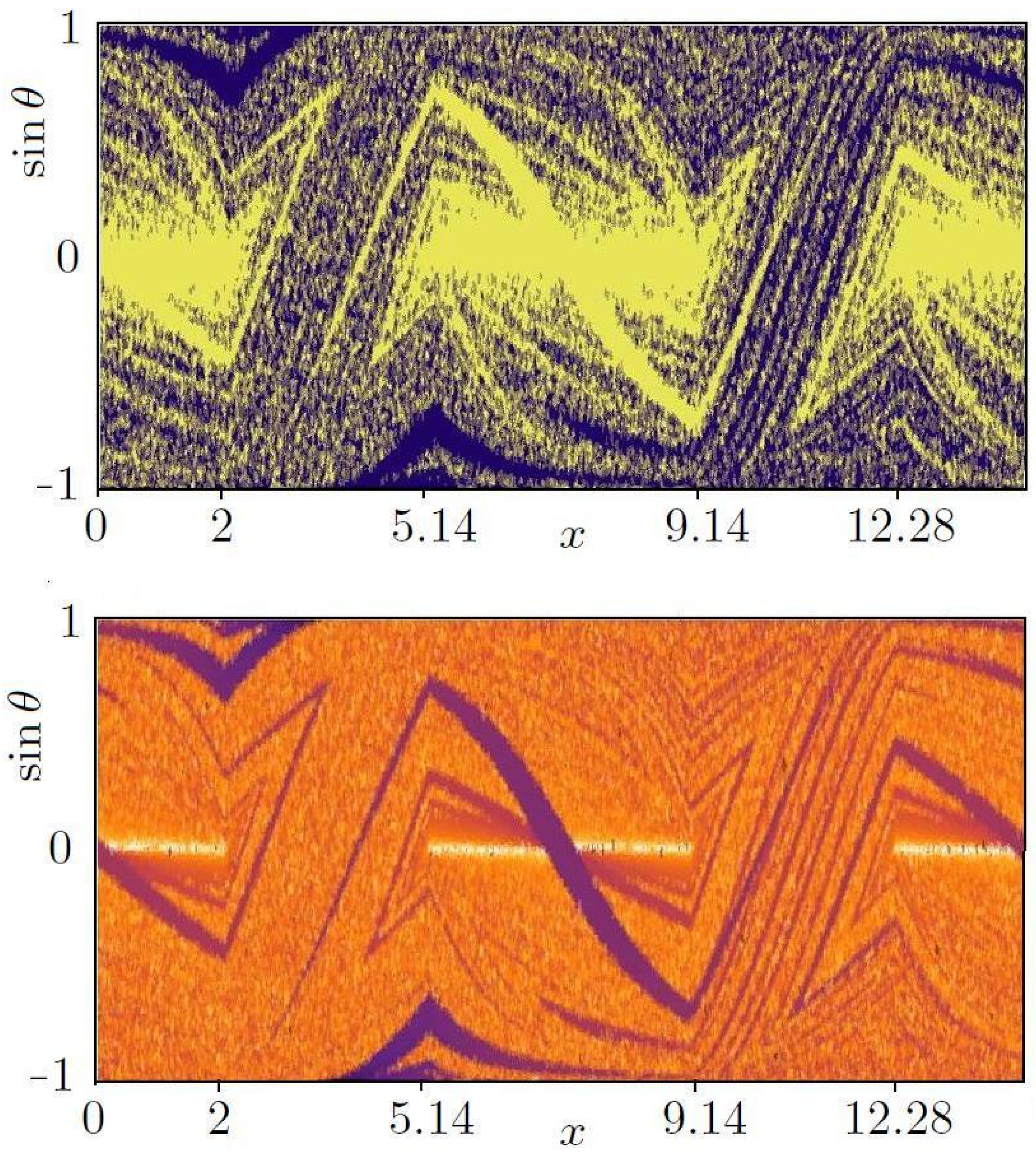


Stadium billiard with two holes H_1 and H_2.

3. Overview of Results

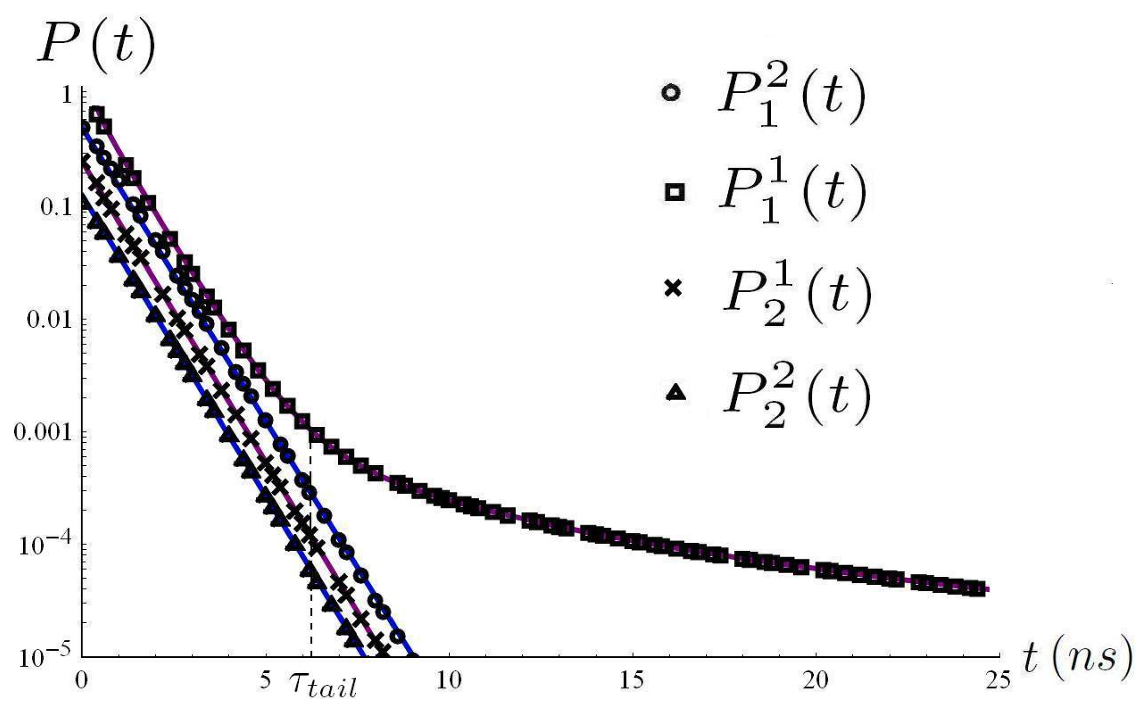
We find that the predominantly chaotic character of the closed stadium is **non-trivially affected by the positioning of holes**. In particular we find that the **transmission and reflection probabilities**, when particles are injected from one of the two holes, are qualitatively different at long times **depending on the choice of the injecting hole, therefore displaying time-dependent asymmetric transport [1]**. The reason for this is that **the stadium's classical phase space is split into separate regions occupied by 'fully-chaotic' and 'sticky' orbits, which are responsible for the exponential and algebraic decays respectively**. As a result the two distributions cannot be treated as time-reverses of each other. Experimental observation in semiconductor nano-structures would imply that the Ehrenfest time and more generally quantum chaos predictions of intermittent system require correction terms subject to the underlying classical dynamics of the corresponding **open** systems.

4. Phase Space Splitting



Top: Initial conditions which escape through $\mathbf{H_1}$ and $\mathbf{H_2}$. **Bottom**: Purple, orange and white corresponds to short, medium and long escape times. ($a = 2, r = 1, h_i = 0.5, h_1^- = 0.25$).

5. Numerical Simulations



$a = 2\ \mu m$, $r = 1\ \mu m$, $h_i = 0.2\ \mu m$ and $h_1^- = 0$. $\tau_{tail} \approx 6.315\ ns$ is the large solution of $e^{-\gamma t} = \frac{D}{\wp_1^1 t^2}$, where $\wp_1^1 \approx 0.5594$.

The power law decay of $P_1^1(t)$ is due to the geometric asymmetry of the hole's positions which exploit the marginally unstable bouncing ball orbits as to force **a preference of escape through** H_1. Furthermore, the splitting of the phase space into fully chaotic and sticky regions renders the later **inaccessible** to particles injected through H_2.

6. Transmission and Reflection survival distributions

Transmission and reflection survival probabilities are defined by $P_i^j(t)$ and $P_i^i(t)$ respectively $(i, j = 1, 2)$, such that $P_i^j(t) = P(x_1 \dots x_{\mathcal{N}} \notin H \big| x_0 \in H_i, x_f \in H_j)$, where $H = H_1 \cup H_2$, $\mathcal{N}(x_0, t)$ is the number of collisions with the boundary up to time t, x_n denotes the position of the particle at the nth collision and x_f is the final (escape) coordinate. We obtain:

$$P_1(t) = e^{-\gamma t} + \frac{D}{t^2} = \wp_1^1 \overbrace{\left(e^{-\gamma t} + \frac{D}{\wp_1^1 t^2}\right)}^{P_1^1(t)} + \wp_1^2 \overbrace{e^{-\gamma t}}^{P_1^2(t)}, \quad (\mathbf{1}) \qquad P_2(t) = e^{-\gamma t} = \wp_2^2 \overbrace{e^{-\gamma t}}^{P_2^2(t)} + \wp_2^1 \overbrace{e^{-\gamma t}}^{P_2^1(t)}, \quad (\mathbf{2})$$

where
$$\gamma = \frac{\sum_{i=1}^k h_i}{\langle\tau\rangle |\partial Q|}, \quad (\mathbf{3}) \qquad \text{and} \qquad D = \frac{r(3\ln 3 + 4)\left((a+h_1^-)^2 + (a-h_1^+)^2\right)}{2h_1 v^2} \quad (\mathbf{4})$$

[2], where $\langle\tau\rangle$ is the mean free path for 2D billiards, $|\partial Q|$ is the perimeter, while the $\wp_i^j$s are the respective asymptotic ($t \to \infty$) reflection and transmission coefficients ($\wp_i^1 + \wp_i^2 = 1$ due to flux conservation, and $\wp_1^2 = \wp_2^1$ due to time-reversal symmetry).

The variety of options available with regards to hole positions, sizes and system parameters offers ways of calibrating and controlling these classical distributions and hence encourages the possibility of experimental observation of the quantum analogue.

7. Quantum time-scales

For typical semiconductor nano-structures, the time scale τ_{tail} at which the algebraic tail becomes visible (see Figure above) is of the order of a nanosecond. This is slightly larger than the predicted Ehrenfest time $\tau_E = \lambda^{-1} \ln\left(\tau_H \langle\tau\rangle / \tau_D^2\right)$ **[3]** (the time scale at which quantum interference effects become apparent $\approx 0.3\ ns$) and would therefore be suppressed. However, since the nature of chaos lies in orbital instability, we argue that **the Ehrenfest time τ_E calculated from the average Lyapunov exponent λ does not faithfully represent quantum spreading of the near-bouncing ball orbits**, since for these orbits the **finite-time local Lyapunov exponent** is zero, leading to a much longer validity and persistence of the classical description of the sticky phase space.

8. Proposed experimental observation

In a stadium quantum-dot, the charge exiting through each hole will follow the driving current, $I_i^{in}(t)$ through hole H_i, with a lag-time τ which is distributed according to (1) or (2) appropriately. This can be modeled by $I_j(t) = (-1)^{i+j} \wp_i^j \int_0^\infty I_i^{in}(t-\tau) \frac{\mathrm{d}P_i^j(\tau)}{\mathrm{d}\tau} \mathrm{d}\tau$. The observed, net current through the system is thus given by $I_i^{net}(t) = I_i^{in}(t) + I_1(t) + I_2(t)$. Because the probability density $\frac{\mathrm{d}P_1^1(\tau)}{\mathrm{d}\tau}$ is slightly skewed to the right, relative to the other densities, the two observables $I_1^{net}(t)$ and $I_2^{net}(t)$ will differ by $\wp_1^1 \int_0^\infty \frac{\mathrm{d}I^{in}(t-\tau)}{\mathrm{d}\tau} \left(P_1^1(t) - P_2^2(t)\right) \mathrm{d}\tau$. A square wave signal $V(t) = V_0 \left(1 + sign(\sin \omega t)\right)$ such that $\omega > \pi/\tau_{tail}$ would thus accentuate the power-law contribution of $P_1^1(t)$ hence assisting in its experimental detection.

9. References

1. C.P. Dettmann and O. Georgiou, *Survival probability for the stadium billiard*, Physica D **238**, 2395, (2009).
2. C.P. Dettmann and O. Georgiou, *Transmission and Reflection in the Stadium Billiard*, Phys.Rev.E, **83**, 036212, (2011).
3. H. Schomerus and P. Jacquod, *Quantum-to-classical correspondence in open chaotic systems*, J. Phys. A: Math. Gen. **38**, 10663, (2005).