Dirichlet process mixtures cribsheet

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Suppose that $\{\theta_i, i = 1, 2, \ldots, n\}$ are random variables in Ω, that G_0 is an arbitrary distribution on Ω, $\alpha > 0$ is a positive real, and that given $\{\theta_i\}$, $\{Y_i, i = 1, 2, \ldots, n\}$ are independent, with $Y_i|\theta \sim f(\cdot|\theta_i)$. The following models are equivalent (in all cases the θ_i are exchangeable):

Ferguson definition of Dirichlet process

The distribution G on Ω is drawn from the Dirichlet process $\mathrm{DP}(\alpha, G_0)$, i.e. for all partitions $\Omega = \bigcup_{j=1}^m B_j$ ($B_j \cap B_k = \emptyset$ if $j \neq k$), and for all m,

$$(G(B_1), \ldots, G(B_m)) \sim \mathrm{Dirichlet}(\alpha G_0(B_1), \ldots, \alpha G_0(B_m))$$

Then, given G, $\{\theta_i\}$ are drawn i.i.d. from G.

Stick-breaking

G is constructed as $\sum_{j=1}^\infty w_j \delta_{\theta_j^\star}$ where $w_j = \prod_{r=1}^{j-1}(1 - V_r)V_j$, $V_r \sim \mathrm{Beta}(1, \alpha)$ are i.i.d. and $\theta_j^\star \sim G_0$ are i.i.d. and independent of $\{V_r\}$. Then given G, $\{\theta_i\}$ are again drawn i.i.d. from G.

Limit of finite mixtures

Draw the Y_i i.i.d. from the finite mixture model $\sum_{j=1}^k w_j f(\cdot|\theta_j^\star)$ where $(w_1, w_2, \ldots, w_k) \sim \mathrm{Dirichlet}(\alpha/k, \alpha/k, \ldots, \alpha/k)$ and $\theta_j^\star \sim G_0$ are i.i.d. and independent of $\{w_j\}$. Then let $k \to \infty$.

A partition model

Partition $\{1, 2, \ldots, n\} = \bigcup_{j=1}^d C_j$ at random so that $p(C_1, C_2, \ldots, C_d) = (\Gamma(\alpha)/\Gamma(\alpha + n)) \alpha^d \prod_{j=1}^d (n_j - 1)!$ where $n_j = \#C_j$. Draw $\theta_j^\star \sim G_0$ i.i.d. for $j = 1, 2, \ldots, d$. For all $i \in C_j$, set $\theta_i = \theta_j^\star$.

Pólya urn representation

Draw $\theta_1 \sim G_0$, and then for all $i = 1, 2, \ldots, n-1$, draw $\theta_{i+1}|\theta_1, \theta_2, \ldots, \theta_i \sim (\alpha/(\alpha + i))G_0 + (1/(\alpha + i)) \sum_{r=1}^i \delta_{\theta_r}$.

Species sampling model (Pólya urn representation using allocations)

Set $z_1 = 1$, then for all $i = 1, 2, \ldots, n-1$, draw $z_{i+1}|z_1, z_2, \ldots, z_i \sim (\alpha/(\alpha + i))\delta_{d_i+1} + (1/(\alpha + i)) \sum_{r=1}^i \delta_{z_r}$ where $d_i = \max\{z_1, z_2, \ldots, z_i\}$. Then draw $\theta_j^\star \sim G_0$ i.i.d., and set $\theta_i = \theta_{z_i}^\star$.

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