MATH11007 NOTES 15: PARAMETRIC CURVES, ARCLENGTH ETC.

1. Parametric representation of curves

The position of a particle moving in three-dimensional space is often specified by an equation of the form

$$\mathbf{x}(t) = (x(t),\, y(t),\, z(t))\ .$$

For instance, take

$$x(t) = \cos t\,,\ \ y(t) = \sin t\,,\ \ z(t) = 0$$

for $t > 0$. We might think of the *parameter* t as "time". As t varies, the particle's position changes, and the curve described by the above equation is the trajectory of the particle. In this particular example, it is obvious that the trajectory is circular; indeed

$$x^2(t) + y^2(t) = 1 \ \text{ for every } t > 0\,.$$

We say that

$$(\cos t\,, \sin t\,, 0)\ ,\ \ 0 \le t < 2\pi\,,$$

is a *parametric representation* of the circle of equation $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

2. Some examples and a bit of MAPLE

Example 2.1.

$$(x(t)\,, y(t)) = (t - 2,\, t/(t - 2))\ ,\ \ t \in \mathbb{R}\,,$$

is a parametric representation of a hyberbola. To see this, we eliminate the parameter t:

$$t = x + 2\,.$$

Hence

$$y = (x + 2)/x = 1 + 2/x\,.$$

Example 2.2. *The curve represented parametrically by*

$$(x(t),\, y(t)) = (t + \sin t\,, 1 - \cos t)\ ,\ \ t \in \mathbb{R}\,,$$

is not so familiar. We can use `MAPLE` *to plot it.* `MAPLE` *is a mathematical software for which the university has a license, and which is installed on the undergraduate computers in the laboratory. Look for it amongst the applications, double-click. When it comes up, select the* `Start with a blank worksheet` *option. At the prompt, type* `?plot[parametric]`. *This explains how to plot parametric curves, and provides some examples. To plot our curve, we use*

```
> plot([t+sin(t),1-cos(t),t=-2*Pi..2*Pi]);
```

The result is shown in Figure 1 (a). The curve is called the cycloid*; it is the curve that describes the trajectory of a point fixed on the rim of a bicycle wheel, as the wheel goes forward.*

Example 2.3. *Consider the curve defined parametrically by*

$$(x(t),\, y(t)) = \big(3t/(1+t^3)\,, 3t^2/(1+t^3)\big)\ ,\quad t \in \mathbb{R}\,.$$

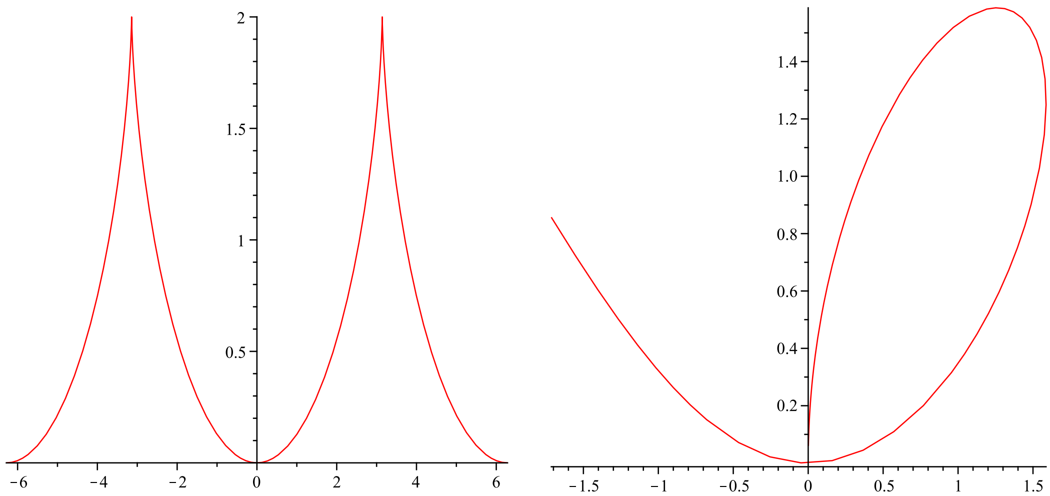
The `MAPLE` *command*

```
> plot([3*t/(1+t^3),3*t^2/(1+t^3),t=-1/2..16*Pi]);
```

produces the plot shown in Figure 1 (b). You can easily verify that

$$x^3 + y^3 = 3xy\,.$$

The curve is called the folium of Descartes*.*



(a) (b)

FIGURE 1. Two famous curves: (a) the cycloid; (b) the folium of Descartes.

3. The tangent line of a curve (in two dimensions) at a point

We have already discussed the problem of determining the line tangent to a curve at a point when the curve can be expressed in the form

$$y = f(x)\,.$$

But now we consider the case where the curve is expressed in parametric form, i.e.

$$(x(t),\, y(t))\ ,\quad t \in [a,b]\,.$$

To find the equation of the line tangent to the curve at, say $t_0 \in [a,b]$, we need to compute

$$\frac{\mathrm{d}y}{\mathrm{d}x}\ \text{ at } (x(t_0),\, y(t_0))\,.$$

We have, by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}y(x(t)) = \frac{\mathrm{d}y}{\mathrm{d}x}(x(t))\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,.$$

So we find

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x(t_0)) = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}(t_0)}{\frac{\mathrm{d}x}{\mathrm{d}t}(t_0)}\,.$$

This result is more neatly expressed if we use the prime symbol to indicate differentiation with respect to x, and the dot symbol to indicate differentiation with respect to t. Then

$$y'(x(t_0)) = \frac{\dot{y}(t_0)}{\dot{x}(t_0)}\,.$$

So the equation for the tangent line is

$$\frac{y - y(t_0)}{x - x(t_0)} = \frac{\dot{y}(t_0)}{\dot{x}(t_0)}\,.$$

The equation of the normal line is easily deduced:

$$\frac{y - y(t_0)}{x - x(t_0)} = -\frac{\dot{x}(t_0)}{\dot{y}(t_0)}\,.$$

Example 3.1. *For the circle*

$$x(t) = \cos t \;\; \textit{and} \;\; y(t) = \sin t\,,$$

the equation of the tangent line at t_0 *is*

$$\frac{y - y(t_0)}{x - x(t_0)} = -\cot t_0\,.$$

The equation of the normal line at t_0 *is*

$$\frac{y - y(t_0)}{x - x(t_0)} = \tan t_0\,.$$

4. Arclength

Consider a curve of equation

$$y = f(x)\,.$$

The length of the infinitesimal arc between the points of horizontal coordinates x and $x + \mathrm{d}x$ is

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}\,.$$

See Figure 2. If the curve is expressed in the parametric form

$$(x(t),\, y(t))\;,\quad a \le t \le b\,,$$

then

$$\mathrm{d}s = \sqrt{\dot{x}^2 + \dot{y}^2}\,\mathrm{d}t\,.$$

We deduce that the length of the curve between $t = a$ and $t = b$ is

$$\int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}\,\mathrm{d}t\,.$$

Example 4.1. *The perimeter of the circle of equation*

$$x(t) = a\cos t\,,\quad y(t) = a\sin t\,,$$

is

$$\int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\,\mathrm{d}t = \int_0^{2\pi} |a|\,\mathrm{d}t = 2\pi|a|\,.$$

Example 4.2. *The curve defined parametrically by*

$$x(t) = a\cos t\,,\quad y(t) = b\sin t\,,$$

is an ellipse. Its perimeter is

$$\int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,\mathrm{d}t\,.$$

It is not straightforward to compute the value of this integral. It may be expressed in terms of the so-called elliptic functions.

5. Curvature

For a curve of equation

$$y = f(x)$$

this is defined as

$$\kappa(x) := \frac{|y''(x)|}{\left\{1 + [y'(x)]^2\right\}^{\frac{3}{2}}}\,. \tag{5.1}$$

See Sheet 3, Q 9.

Let us work out the curvature when the curve is expressed in the parametric form

$$(x(t),\, y(t))\,.$$

We have shown earlier that

$$y' = \frac{\dot{y}}{\dot{x}}\,.$$

Hence, by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}y' = y''\dot{x}\,,$$

and so

$$y'' = \frac{\frac{\mathrm{d}}{\mathrm{d}t}y'}{\dot{x}} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{y}}{\dot{x}}}{\dot{x}} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^3}\,.$$

Hence the "parametric form" of the curvature (5.1) is

$$\kappa(t) = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{[\dot{x}^2 + \dot{y}^2]^{\frac{3}{2}}}\,.$$

Example 5.1. *For the ellipse of equation*

$$x(t) = a\cos t \quad \text{and} \quad y(t) = b\sin t$$

we find

$$\kappa(t) = \frac{\left|ab\sin^2 t + ab\cos^2 t\right|}{\left[a^2\sin^2 t + b^2\cos^2 t\right]^{\frac{3}{2}}} = \frac{|ab|}{\left[a^2\sin^2 t + b^2\cos^2 t\right]^{\frac{3}{2}}}\,.$$

In particular, for the circle, $a = b$ and we obtain the expected result

$$\kappa(t) = \frac{1}{a}\,.$$

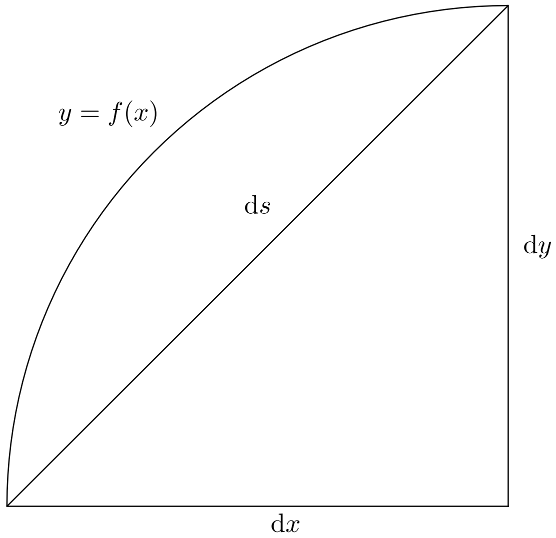


FIGURE 2. The length of an infinitesimal arc of the curve $y = f(x)$ is $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$

REFERENCES

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, McGraw–Hill, 1999.
2. E. Hairer and G. Wanner, *Analysis by its History*, Springer-Verlag, New-York, 1996.