MATH11007 NOTES 16: INTRODUCTION TO SEVERAL VARIABLES

1. Outline

So far, the functions we have considered have been defined on a subset of the real numbers. We shall now begin our study of functions of several variables— that is, functions whose domain is a subset of $\mathbb{R}^d$, where $d \geq 1$. Recall what we learnt in the particular case $d = 1$. We considered the notions of *function*, of *limit*, of *differentiability* and of *integral*. We will look at each of these notions again, and see how they can be generalised in the context of several variables. Those who have mastered the one-variable case will have no great difficulty in dealing with the more general case; with some exaggeration, the main hurdle is the notation, i.e. the complexity of dealing with several variables at once.

2. Some notation and terminology

If $d \in \mathbb{N}$ is unspecified, we write

$$\mathbf{x} = (x_1,\, x_2,\, \ldots,\, x_d) \in \mathbb{R}^d$$

for a generic point in d-dimensional space. Some special cases (with a special notation):

(1) $d = 1$: we write $\mathbf{x} = x$.
(2) $d = 2$: we write $\mathbf{x} = (x,\, y)$.
(3) $d = 3$: we write $\mathbf{x} = (x,\, y,\, z)$.

In some physical applications, we might consider variables consisting of space and time. Then

$$\mathbf{x} = (x,\, y,\, z,\, t)\,.$$

It is straightforward to extend to several variables the notions of *function* and of *maximal domain*: a (real-valued) function of several variables consist of (a) a domain, say $A \subseteq \mathbb{R}^d$, (b) a codomain, say $B \subseteq \mathbb{R}$, and (c) a rule or mapping, say f, that associates to every element of the domain a unique element of the codomain. The maximal domain of a function f is the largest subset of $\mathbb{R}^d$ for which the mapping makes sense. The *range* of a function $f : A \mapsto B$ is the set

$$\{f(\mathbf{x}) : \mathbf{x} \in A\} \subseteq B\,.$$

Example 2.1. *Consider the mapping*

$$f(x, y) = \ln\left(y^2 - 4x + 8\right)\,.$$

The formula makes sense if and only if $y^2 - 4x + 8 > 0$, *i.e. if and only if*

$$x < \frac{y^2}{4} + 2\,.$$

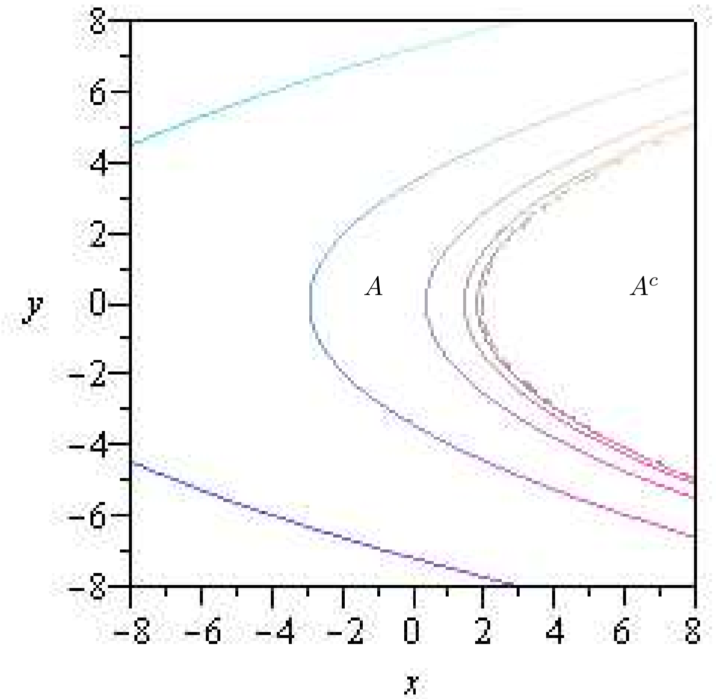

FIGURE 1. Level curves of the function $f(x, y) = \ln\bigl(y^2 - 4x + 8\bigr)$. The region A is in the maximal domain of f; the region A^c is not.

3. Level curves

Let $d \geq 2$, $c \in \mathbb{R}$ and $f : A \mapsto \mathbb{R}$. The points of $\mathbb{R}^d$ satisfying the equation

$$f(\mathbf{x}) = c$$

form a *hypersurface* in $\mathbb{R}^d$. We call these hypersurfaces the *level hypersurfaces* of the function f; they are parametrised by the number c. When $d = 2$, this "hypersurface" is simply a curve in the (x, y) plane, and we speak of the *level curves* of the function $f = f(x, y)$.

Example 3.1. *The level curves of the function defined in the previous example are the curves of equation*

$$x = \frac{y^2}{4} + 2 - \frac{\mathrm{e}^{-c}}{4}\,.$$

These are the parabolae shown in Figure 1. The boundary between the maximal domain and its complement is obtained by taking $c = -\infty$.

Example 3.2. *People with an interest in outdoor pursuits will be familiar with Ordnance Survey maps. These maps show the level curves*

$$z = f(x, y)$$

where z is the height above sea level at position (x, y). These level curves give a good feel for the topography of the mapped region. In particular, where the terrain is steep, one finds that the level curves are closely spaced.

Returning to the previous example, we see a clustering of the level curves near the parabola of equation $x = \frac{y^2}{4} + 2$. This is a manifestation of the fact that the function becomes unbounded there.

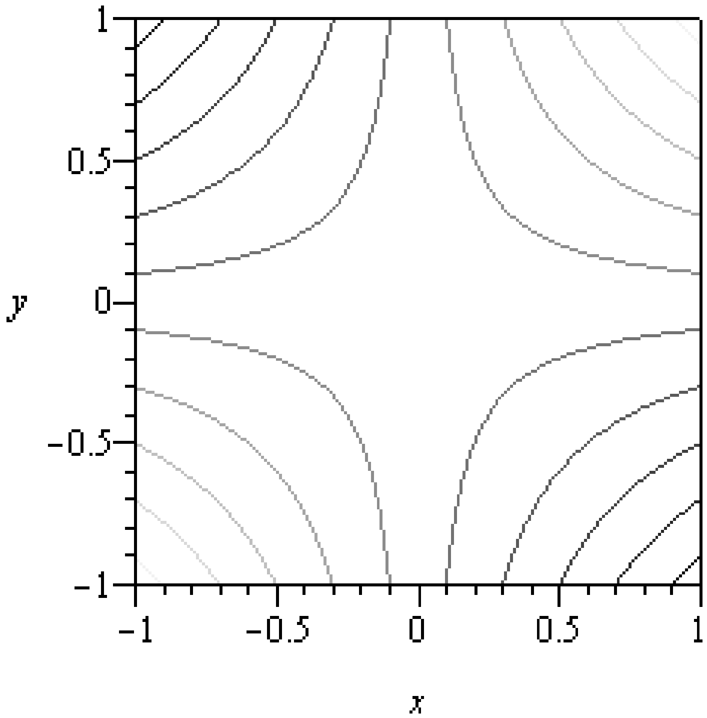

FIGURE 2. Level curves of the function $f(x,y) = xy$. The maximal domain is the whole of $\mathbb{R}^2$.

Example 3.3. *The level curves of the function*

$$f(x,y) = xy$$

are the hyperbolae of equation

$$xy = c\,.$$

The `Maple` *command*

```
> plot3d(x*y,x=-1..1,y=-1..1,style=contour,orientation=[-90,0],axes=boxed);
```

produces the plot shown in Figure 2. The level curves of the function

$$g(x,y) = y/x \quad x \neq 0,$$

are the straight lines of equation

$$y = cx\,.$$

The maximal domain is $\mathbb{R}^2 \backslash \{(0,y) : y \in \mathbb{R}\}$*. This is a particularly simple example, since straight lines are very familiar curves! Use it to reflect on the concept of level curve: note that, for each choice of c, we get a line of slope c through the origin. Obviously, one cannot draw* all *the level curves. But by picking some values of c and drawing the corresponding levek curves, we gain some insight into the function's behaviour.*

4. The notion of limit

Let us discuss very briefly how to extend the concept of *limit* to functions of several variables. Recall the definition we used in the one-variable case: for a function $f : A \mapsto \mathbb{R}$ and x_0 either an element of A or an accumulation point,

$$\lim_{x \to x_0} f(x) = L$$

means

$$\forall\, \varepsilon > 0\,,\ \exists\, \delta > 0 \text{ such that } x \in A \text{ and } |x - x_0| < \delta \implies |f(x) - L| < \varepsilon\,.$$

There is no great difficulty in extending this definition; all we need is a measure of *distance* between two points in $\mathbb{R}^d$. To this end, we use the *norm*

$$\|\mathbf{x}\| := \left(\sum_{j=1}^{d} x_j^2 \right)^{\frac{1}{2}}\,.$$

This an obvious generalisation to d variables of the familiar absolute value of a number. Then we say that

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = L \tag{4.1}$$

if

$$\forall\, \varepsilon > 0\,,\ \exists\, \delta > 0 \text{ such that } \mathbf{x} \in A \text{ and } \|\mathbf{x} - \mathbf{x}_0\| < \delta \implies |f(\mathbf{x}) - L| < \varepsilon\,.$$

There is however, a significant difference between one and more variables: in the one variable case, a limit can be approached from at most two directions, namely from the right or from the left. In the many-variable case, a limit can be approached in infinitely many ways. We state without proof that, if a number L exits such that (4.1) holds, then this number is unique, and so cannot depend on the way in which $\mathbf{x}$ approaches $\mathbf{x}_0$.

Example 4.1. *Does the limit*

$$\lim_{(x,y) \to (0,0)} \frac{y}{x}$$

exist? No. To prove this, let (x, y) *approach* $(0, 0)$ *along the line of equation* $y = mx$. *Then*

$$\frac{y}{x} = \frac{mx}{x} \xrightarrow[x \to 0]{} m\,.$$

Since the right-hand side depends on m*, the limit cannot exist.*

5. Partial differentiation

In the case $d = 1$, we defined the derivative of a function f at a point x as the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\,.$$

The extension of this concept to several variable is *not* trivial: it makes no sense to divide two points in $\mathbb{R}^d$. The correct way of defining the derivative will be investigated in the Calculus 2 unit. Here, we settle for a weaker concept.

Definition 5.1. Let $A \subseteq \mathbb{R}^d$ and $f : A \mapsto \mathbb{R}$. The *partial derivative* of f with respect to x_i at $\mathbf{x}$ (if it exists) is the limit

$$\lim_{\mathbb{R} \ni h \to 0} \frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x})}{h}$$

where $\mathbf{e}_i$ is the point of $\mathbb{R}^d$ with ith coordinate 1 and every other coordinate 0.

This partial derivative is denoted

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) \quad \text{or} \quad f_{x_i}(\mathbf{x})\,.$$

We emphasise that the limit in the definition is a "one-variable" limit; all the variables except the ith are treated as constants.

Example 5.1. *Let*

$$f(x,y) = \mathrm{e}^{x^2+y}\,.$$

Using the definition,

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{\mathrm{e}^{y+(x+h)^2} - \mathrm{e}^{y+x^2}}{h} = \mathrm{e}^{y} \lim_{h\to 0} \frac{\mathrm{e}^{(x+h)^2} - \mathrm{e}^{x^2}}{h}$$
$$= \mathrm{e}^{y}\, 2x\, \mathrm{e}^{x^2} = 2x\, \mathrm{e}^{x^2+y}\,.$$

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{\mathrm{e}^{(y+h)+x^2} - \mathrm{e}^{y+x^2}}{h} = e^{x^2} \lim_{h\to 0} \frac{\mathrm{e}^{(y+h)} - \mathrm{e}^{y}}{h}$$
$$= \mathrm{e}^{x^2}\, \mathrm{e}^{y} = \mathrm{e}^{x^2+y}\,.$$

Alternatively, to calculate f_x, we treat y as a constant. Then

$$\frac{\partial f}{\partial x}(x,y) = \mathrm{e}^{y} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{x^2} = \mathrm{e}^{y}\, 2x\, \mathrm{e}^{x^2} = 2x\, \mathrm{e}^{y+x^2}\,.$$

To calculate f_y, we treat x as a constant:

$$\frac{\partial f}{\partial y}(x,y) = \mathrm{e}^{x^2} \frac{\mathrm{d}}{\mathrm{d}y} \mathrm{e}^{y} = \mathrm{e}^{x^2}\, \mathrm{e}^{y} = \mathrm{e}^{y+x^2}\,.$$

Example 5.2. *Let*

$$f(x,y,z) = xy^2z^3\,.$$

Then

$$\frac{\partial f}{\partial x}(x,y,z) = \lim_{h\to 0} \frac{(x+h)y^2z^3 - xy^2z^3}{h} = y^2z^3 \lim_{h\to 0} \frac{(x+h)-x}{h} = y^2z^3$$
$$= y^2z^3 \frac{\mathrm{d}}{\mathrm{d}x} x\,.$$

$$\frac{\partial f}{\partial y}(x,y,z) = \lim_{h\to 0} \frac{x(y+h)^2z^3 - xy^2z^3}{h} = xz^3 \lim_{h\to 0} \frac{(y+h)^2 - y^2}{h} = xz^3 2y$$
$$= xz^3 \frac{\mathrm{d}}{\mathrm{d}y} y^2\,.$$

$$\frac{\partial f}{\partial z}(x,y,z) = \lim_{h\to 0} \frac{xy^2(z+h)^3 - xy^2z^3}{h} = xy^2 \lim_{h\to 0} \frac{(z+h)^3 - z^3}{h} = xy^2 3z^2$$
$$= xy^2 \frac{\mathrm{d}}{\mathrm{d}z} z^3\,.$$

6. Higher partial derivatives

Let $A \subseteq \mathbb{R}^d$ and $f : A \mapsto \mathbb{R}$. We have defined and illustrated the first partial derivatives of f:

$$\frac{\partial f}{\partial x_i}(\mathbf{x}), \quad 1 \leq i \leq d.$$

If we let $\mathbf{x}$ vary, and the partial derivative $f_{x_i}(\mathbf{x})$ with respect to x_i exists for all $\mathbf{x} \in A$, then f_{x_i} is a function of $\mathbf{x}$ with domain A. So we may consider the partial derivatives of f_{x_i}. We use the notation

$$\frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i} = \frac{\partial^2 f}{\partial x_j \partial x_i}, \quad 1 \leq i, j \leq d,$$

or

$$f_{x_i x_j}, \quad 1 \leq i, j \leq d.$$

These are called the *second partial derivatives* of f.

Example 6.1. *Let*

$$f(x, y) = \sin\left(xy^2\right).$$

Then

$$f_x = y^2 \cos\left(xy^2\right) \quad \textit{and} \quad f_y = 2xy \cos\left(xy^2\right).$$

So

$$f_{xx} = -y^4 \sin\left(xy^2\right), \quad f_{xy} = 2y\cos\left(xy^2\right) - 2xy^3 \sin\left(xy^2\right),$$
$$f_{yx} = 2y\cos\left(xy^2\right) - 2xy^3 \sin\left(xy^2\right), \quad f_{yy} = 2x\cos\left(xy^2\right) - 4x^2y^2 \sin\left(xy^2\right).$$

It may be noted that, in this example,

$$f_{xy} = f_{yx}.$$

This is true more generally as long as f is "nice enough":

$$\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_j \partial x_i}.$$

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, McGraw–Hill, 1999.
2. E. Hairer and G. Wanner, *Analysis by its History*, Springer-Verlag, New-York, 1996.