MATH11007 NOTES 19: LOCAL EXTREMA IN SEVERAL VARIABLES.

1. Local extrema in one variable

Every student of calculus knows that the concept of derivative is exceedingly useful when optimising smooth functions of one variable. Let us review (once more!) some familiar facts.

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, then x_0 is a *critical point* of f if

$$f'(x_0) = 0\,.$$

Furthermore, all the local extrema of f are critical points.

The critical points are classified as follows:

(1) If $f''(x_0) < 0$ then x_0 is a local maximum.
(2) If $f''(x_0) = 0$ then x_0 is an inflexion point.
(3) If $f''(x_0) > 0$ then x_0 is a local minimum.

Our purpose this week is to examine how these well-known results may be extended to the case where the function f depends on *several* variables.

2. Critical points and local extrema

Our strategy for studying the multivariable case is to work—as far as possible—with useful functions of one variable. First, however, in order to have a clear discussion, it is necessary to introduce a minimum(!) of definitions and terminology. Let $f : \mathbb{R}^d \to \mathbb{R}$ have first partial derivatives. We say that $\mathbf{x}_0$ is a *critical point* of f if

$$\nabla f(\mathbf{x}_0) = \mathbf{0}\,.$$

We say that $\mathbf{x}_0$ is a *local minimum* of f if there exists $\varepsilon > 0$ such that

$$\forall\, \mathbf{x} \in \mathbb{R}^d,\ \ \|\mathbf{x} - \mathbf{x}_0\| < \varepsilon \implies f(\mathbf{x}) \ge f(\mathbf{x}_0)\,.$$

We say that $\mathbf{x}_0$ is a *local maximum* of f if there exists $\varepsilon > 0$ such that

$$\forall\, \mathbf{x} \in \mathbb{R}^d,\ \ \|\mathbf{x} - \mathbf{x}_0\| < \varepsilon \implies f(\mathbf{x}) \le f(\mathbf{x}_0)\,.$$

We say that $\mathbf{x}_0$ is a local extremum of f if it is either a local maximum or a local minimum.

The following theorem generalises the familiar one-variable result.

Theorem 2.1. *If $\mathbf{x}_0$ is a local extremum of f then it is a critical point of f.*

Proof. Suppose that $\mathbf{x}_0$ is a local extremum. Let $\mathbf{u}$ be a direction in $\mathbb{R}^d$ and consider the curve of parametric equation

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{u}\,,\ \ t \in \mathbb{R}\,.$$

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Set

$$\varphi(t) = f(\mathbf{x}(t))\,.$$

Then φ is a function of just one variable, and $t = 0$ is a local extremum of φ. Hence

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(0) = \nabla f(\mathbf{x}_0) \cdot \mathbf{u}\,.$$

Now, $\mathbf{u}$ is an arbitrary direction. It follows (how?) that

$$\nabla f(\mathbf{x}_0) = \mathbf{0}\,.$$

□

3. Classification of the critical points

Next, we would like to be able to classify the critical points in a manner analogous to the one-variable case. Let $\mathbf{x}_0$ be a critical point, choose an arbitrary direction $\mathbf{u}$, and consider again the function

$$\varphi(t) = f(\mathbf{x}(t))\,, \quad \mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{u}\,.$$

We have

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\nabla f(\mathbf{x}(t)) \cdot \mathbf{u}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial f}{\partial x_1}(\mathbf{x}(t))u_1 + \cdots + \frac{\partial f}{\partial x_d}(\mathbf{x}(t))u_d\right]\,.$$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\frac{\partial f}{\partial x_j}}_{F}(\mathbf{x}(t))u_j = u_j\frac{\mathrm{d}}{\mathrm{d}t}F(\mathbf{x}(t)) = u_j\nabla F(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}$$

$$= u_j\nabla F(\mathbf{x}(t)) \cdot \mathbf{u} = u_j\nabla\frac{\partial f}{\partial x_j}(\mathbf{x}(t)) \cdot \mathbf{u}\,.$$

Hence

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}(t) = \left[u_1\nabla\frac{\partial f}{\partial x_1}(\mathbf{x}(t)) \cdot \mathbf{u} + \cdots + u_d\nabla\frac{\partial f}{\partial x_d}(\mathbf{x}(t)) \cdot \mathbf{u}\right]\,.$$

This complicated-looking formula may be expressed neatly in matrix form:

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}(t) = \mathbf{u}^T H_f(\mathbf{x}(t))\mathbf{u} \tag{3.1}$$

where

$$H_f := \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_d} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^2 f}{\partial x_d \partial x_1} & \frac{\partial^2 f}{\partial x_d \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_d^2} \end{pmatrix} \tag{3.2}$$

is the *Hessian matrix* of f. This is a matrix-valued function of $\mathbf{x}$ which plays the part of the second derivative.

In the one-variable case, the classification of the critical points was based on the sign of the second derivative. We need a concept of "sign" for matrices ...

Definition 3.1. We say that a $d \times d$ matrix A is

(1) *positive definite* if

$$\forall \mathbf{x} \in \mathbb{R}^d\,, \quad \mathbf{x} \neq \mathbf{0} \implies \mathbf{x}^T A\mathbf{x} > 0\,;$$

(2) *negative definite* if

$$\forall \mathbf{x} \in \mathbb{R}^d\,, \quad \mathbf{x} \neq \mathbf{0} \implies \mathbf{x}^T A \mathbf{x} < 0\,;$$

(3) *indefinite* if

$$\exists\, \mathbf{x} \neq \mathbf{0},\, \mathbf{y} \neq \mathbf{0} \in \mathbb{R}^d\,, \quad \text{such that } \mathbf{x}^T A \mathbf{x} > 0 \text{ and } \mathbf{y}^T A \mathbf{y} < 0\,.$$

We are now ready to characterise the local extrema of f in terms of the Hessian. Suppose for example that the Hessian of f at $\mathbf{x}_0$ is positive definite. Then

$$\mathbf{u}^T H_f(\mathbf{x}_0)\mathbf{u} > 0$$

and

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}(0) > 0$$

must hold for every direction $\mathbf{u}$. Hence $t = 0$ is a local minimum of φ no matter what the direction $\mathbf{u}$. With a little work, we can deduce that $\mathbf{u}$ is a local minimum of f. An analogous statement can be made if the Hessian is, instead, negative definite. The situation may be summarised as follows:

(1) If $H_f(\mathbf{x}_0)$ is positive definite, then the critical point $\mathbf{x}_0$ is a local minimum of f.
(2) If $H_f(\mathbf{x}_0)$ is indefinite, then the critical point $\mathbf{x}_0$ is not a local extremum; it is called a *saddle point* of f.
(3) If $H_f(\mathbf{x}_0)$ is negative definite, then the critical point $\mathbf{x}_0$ is a local maximum of f.

4. The eigenvalues of the Hessian

This classification is all very well, but how can one test for definiteness in practice? You will— very shortly— encounter the following concept in the Linear Algebra unit:

Definition 4.1. We say that the complex number λ is an *eigenvalue* of the $d \times d$ matrix A if there exists a non-zero vector $\mathbf{x} \in \mathbb{C}^d$ such that

$$A\mathbf{x} = \lambda \mathbf{x}\,.$$

It is in principle straightforward to calculate the eigenvalues, given the matrix A: they are the roots of the *characteristic polynomial* of A, i.e. they satisfy

$$|A - \lambda I| = 0\,,$$

where the vertical bars denote the determinant, and I is the $d \times d$ identity matrix.

The following theorem makes the connection with our topic clear.

Theorem 4.1. *The $d \times d$ real symmetric matrix A is*

(1) *positive-definite if and only if all its eigenvalues are strictly positive;*
(2) *indefinite if and only if it has eigenvalues of both signs;*
(3) *negative-definite if and only if all its eigenvalues are strictly negative.*

Example 4.1. *Let $a \neq 0$ and $b \neq 0$. Find and classify the critical points of*

$$f(x, y) = a(x-1)^2 + by^2\,.$$

Solution: *We have*

$$\nabla f(x, y) = 2(a(x-1),\, by)\,.$$

Hence the only critical point of f *is* $(x_0,\, y_0) = (1, 0)$. *Obviously,*

$$H_f(x_0, y_0) := \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x_0, y_0) & \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\ \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) & \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}.$$

The eigenvalues λ *satisfy*

$$\begin{vmatrix} 2a - \lambda & 0 \\ 0 & 2b - \lambda \end{vmatrix} := (2a - \lambda)(2b - \lambda) = 0\,.$$

Hence the eigenvalues are $2a$ *and* $2b$. *We deduce that*

(1) *If* a *and* b *are both positive, then the critical point is a local minimum.*
(2) *If* a *and* b *are both negative, then the critical point is a local maximum.*
(3) *If* $ab < 0$, *then the critical point is a saddle.*

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, McGraw–Hill, 1999.
2. E. Hairer and G. Wanner, *Analysis by its History*, Springer-Verlag, New-York, 1996.