MATH11007 NOTES 20: INTEGRATION OVER TWO-DIMENSIONAL DOMAINS.

1. Introduction

We consider the problem of computing the volume enclosed between the region D in the xy-plane and the surface of equation $z = f(x,y)$; see Figure 1. We denote this volume by

$$\iint_D f(x,y)\,\mathrm{d}x\mathrm{d}y\,. \tag{1.1}$$

This is considerably harder than the calculation of the area under a given curve. Indeed, for *multiple integrals*, i.e. integrals involving more than one variable of integration, there is no analog of the Fundamental Theorem of Calculus that was so useful in the one-variable case. The general strategy for computing these multiple integrals will be to reduce them to the computation of a succession of single integrals.

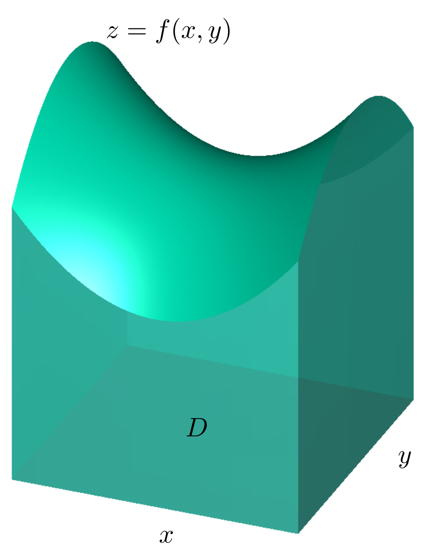

Figure 1. The volume under the surface $z = x^2 - y^2$. The rectangular region D at the bottom is the domain of integration for the corresponding double integral, while the surface is the graph of the two-variable integrand [3].

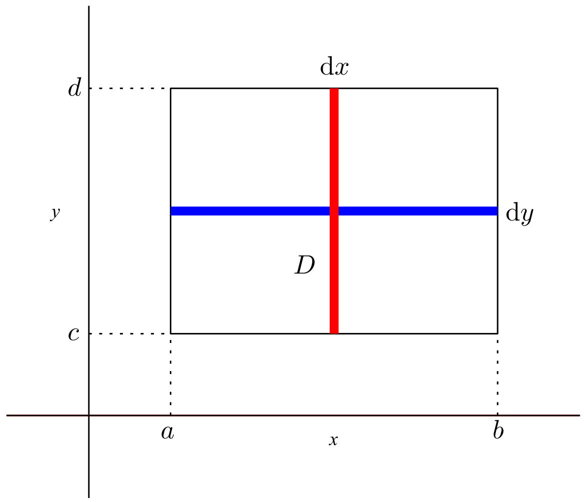

Figure 2. A rectangular domain of integration in the xy-plane. The domain can be "sliced" horizontally or vertically.

We will not attack the problem in its full generality. Rather, we shall start from the simplest situation, and build up our skills by considering progressively more challenging cases.

2. Rectangular domains

The computation of the double integral is simplest when D has a rectangular shape, as shown in Figure 2.

To motivate the approach with a concrete analogy, imagine that you are given the task of computing the volume of a loaf of bread, but that your expertise is really in computing areas. Think of Figure 2 as showing a view of the loaf directly from above. One strategy for computing the volume is to cut the loaf into thin slices. The volume of each slice is the product of its surface area and its thickness. For instance, consider a horizontal slice for a fixed value of y of width $\mathrm{d}y$. The corresponding slice of the volume under the surface $z = f(x, y)$ has the value

$$[\text{area under the curve } z = f(x, y) \text{ for } y \text{ fixed}] \times \mathrm{d}y = \int_a^b f(x, y)\, \mathrm{d}x\mathrm{d}y\,.$$

The total volume is obtained by summing the contributions from all these infinitesimal horizontal slices:

$$\iint_D f(x, y)\, \mathrm{d}x\mathrm{d}y = \int_c^d \left\{ \int_a^b f(x, y)\, \mathrm{d}x \right\} \mathrm{d}y\,. \tag{2.1}$$

We have thus computed the volume by summing over horizontal slices, but why not cut *vertical* slices instead; see Figure 2 again. This time the volume of a slice is

$$[\text{area under the curve } z = f(x, y) \text{ for } x \text{ fixed}] \times \mathrm{d}x = \int_c^d f(x, y)\, \mathrm{d}y\mathrm{d}x\,.$$

This gives another formula for the total volume:

$$\iint_D f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_a^b \left\{ \int_c^d f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x\,. \tag{2.2}$$

These two formulae should give the same result:

$$\int_c^d \left\{ \int_a^b f(x,y)\,\mathrm{d}x \right\} \mathrm{d}y = \int_a^b \left\{ \int_c^d f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x\,. \tag{2.3}$$

We express this fact in words by saying that "one can change the order of integration without changing the result".

Example 2.1. *Take*

$$f(x,y) = \mathrm{e}^{x+2y}\,.$$

"Horizontal slicing" gives

$$\begin{aligned}\int_c^d \left\{ \int_a^b \mathrm{e}^{x+2y}\,dx \right\} dy &= \int_c^d \mathrm{e}^{2y} \left[\mathrm{e}^x \Big|_a^b \right] dy = \int_c^d \mathrm{e}^{2y} \left[\mathrm{e}^b - \mathrm{e}^a \right] dy \\ &= \left[\mathrm{e}^b - \mathrm{e}^a \right] \int_c^d \mathrm{e}^{2y}\,dy = \left[\mathrm{e}^b - \mathrm{e}^a \right] \frac{1}{2}\mathrm{e}^{2y} \Big|_c^d \\ &= \frac{1}{2} \left[\mathrm{e}^b - \mathrm{e}^a \right] \left[\mathrm{e}^{2d} - \mathrm{e}^{2c} \right]\,.\end{aligned}$$

On the other hand, "vertical slicing" gives

$$\begin{aligned}\int_a^b \left\{ \int_c^d \mathrm{e}^{x+2y}\,dy \right\} dx &= \int_a^b \mathrm{e}^x \left[\frac{1}{2}\mathrm{e}^{2y} \Big|_c^d \right] dx = \int_a^b \mathrm{e}^x \frac{1}{2} \left[\mathrm{e}^{2d} - \mathrm{e}^{2c} \right] dx \\ &= \frac{1}{2} \left[\mathrm{e}^{2d} - \mathrm{e}^{2c} \right] \int_a^b \mathrm{e}^x\,dx = \frac{1}{2} \left[\mathrm{e}^{2d} - \mathrm{e}^{2c} \right] \mathrm{e}^x \Big|_a^b \\ &= \frac{1}{2} \left[\mathrm{e}^{2d} - \mathrm{e}^{2c} \right] \left[\mathrm{e}^b - \mathrm{e}^a \right]\,.\end{aligned}$$

So the result is the same in both cases.

3. Triangular domains

The next step up in complexity is to consider the case where the domain has a triangular shape. For definiteness, we suppose that the vertices are

$$(0,0)\,,\ \ (a,0)\ \text{ and }\ (a,b)\,.$$

There is no real loss of generality in this assumption because any triangle can be mapped to the triangle with these particular vertices. (We shall study next week how to use "substitutions" in double integrals.) The resulting triangular domain is shown in Figure 3.

Horizontal slicing. Using the same strategy as in the rectangular case, let us slice this domain D horizontally. The particular slice corresponding to the value $y \in [0,b]$ runs from a point on the hypothenuse — the edge joining the vertices $(0,0)$ and (a,b)— of the triangle to a point on the vertical edge joining the vertices $(a,0)$ and (a,b). Now the equation of the line along the hypothenuse is

$$y = \frac{b}{a}x\,.$$

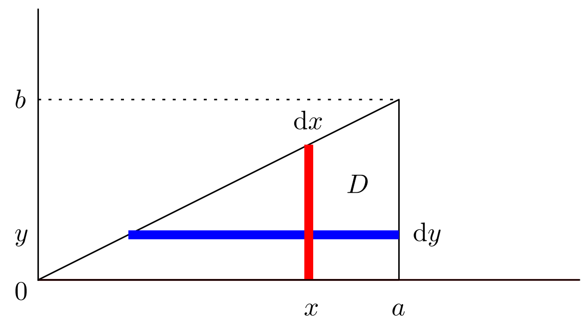

FIGURE 3. A triangular domain of integration in the xy-plane. The domain can again be "sliced" horizontally or vertically, but the limits of integration will depend on the choice made.

So, for the slice corresponding to y, the variable x runs from ay/b to a. The volume corresponding to a horizontal slice of infinitesimal width $\mathrm{d}y$ is therefore

$$[\text{area under the curve } z = f(x,y) \text{ for } y \text{ fixed}] \times \mathrm{d}y = \int_{\frac{ay}{b}}^{a} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

and the total volume is

$$\int_0^b \left\{ \int_{\frac{ay}{b}}^{a} f(x,y)\,\mathrm{d}x \right\} \mathrm{d}y\,. \tag{3.1}$$

Vertical slicing. We could equally well slice the domain vertitally. The particular slice corresponding to the value $x \in [0,a]$ runs from the point $(x,0)$ to the point $(x, \frac{b}{a}x)$ on the hypothenuse of the triangle; see Figure 3. The volume corresponding to this vertical slice of infinitesimal width $\mathrm{d}x$ is therefore

$$[\text{area under the curve } z = f(x,y) \text{ for } x \text{ fixed}] \times \mathrm{d}x = \int_0^{\frac{bx}{a}} f(x,y)\,\mathrm{d}y\mathrm{d}x$$

and the total volume is

$$\int_0^a \left\{ \int_0^{\frac{bx}{a}} f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x\,. \tag{3.2}$$

Whatever the order of integration, the result should be the same:

$$\iint_D f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_0^b \left\{ \int_{\frac{ay}{b}}^{a} f(x,y)\,\mathrm{d}x \right\} \mathrm{d}y = \int_0^a \left\{ \int_0^{\frac{bx}{a}} f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x\,.$$

The novelty in the triangular case is that, if we change the order of integration, we have to adjust the limits of integration accordingly.

Example 3.1. *Take*

$$a = b = 1$$

and

$$f(x,y) = x - y^2 \,.$$

By Equation (3.1), if we slice horizontally, we find

$$\begin{aligned}\iint_D x - y^2 \, dx\,dy &= \int_0^1 \left\{ \int_y^1 x - y^2 \, dx \right\} \, dy = \int_0^1 \left[\frac{x^2}{2} - xy^2 \right] \Big|_y^1 \, dy \\ &= \int_0^1 \left[\frac{1}{2} - \frac{3}{2} y^2 + y^3 \right] \, dy = \left[\frac{y}{2} - \frac{1}{2} y^3 + \frac{1}{4} y^4 \right] \Big|_0^1 = \frac{1}{4} \,.\end{aligned}$$

On the other hand, if we slice horizontally, Equation (3.2) gives

$$\begin{aligned}\iint_D x - y^2 \, dx\,dy &= \int_0^1 \left\{ \int_0^x x - y^2 \, dy \right\} \, dx = \int_0^1 \left[xy - \frac{y^3}{3} \right] \Big|_0^x \, dx \\ &= \int_0^1 \left[x^2 - \frac{x^3}{3} \right] \, dx = \left[\frac{x^3}{3} - \frac{x^4}{12} \right] \Big|_0^1 = \frac{1}{4} \,,\end{aligned}$$

as it should.

The *value* of the double integral does not depend on the order of integration. But choosing the “right” order can make its computation easier.

Example 3.2. *Take the same triangular domain as in the previous example, i.e. D is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$, and compute the integral*

$$\iint_D y\sqrt{1+x^3} \, dx\,dy \,.$$

Using horizontal slicing, this integral equals

$$\int_0^1 \left\{ \int_y^1 y\sqrt{1+x^3} \, dx \right\} \, dy \,.$$

But we are unable to find a primitive for the inner integral.

On the other hand, vertical slicing gives

$$\begin{aligned}\int_0^1 \left\{ \int_0^x y\sqrt{1+x^3} \, dy \right\} \, dx &= \int_0^1 \left[\frac{y^2}{2} \sqrt{1+x^3} \right] \Big|_0^x \, dx = \int_0^1 \frac{x^2}{2} \sqrt{1+x^3} \, dx \\ &\overset{\substack{u=1+x^3 \\ \downarrow}}{=} \frac{1}{6} \int_1^2 \sqrt{u} \, du = \frac{1}{9} u^{\frac{3}{2}} \Big|_1^2 = \frac{2\sqrt{2}-1}{9} \,.\end{aligned}$$

4. More general domains

Having tried our hand on some particular cases, we can now attempt a more general domain. Suppose for instance that the domain D of the integrand f is of the form

$$D = \{(x,y) : \, a \le x, \; L(x) \le y \le U(x)\} \,.$$

See Figure 4. For such a domain, vertical slicing is appropriate:

$$\iint_D f(x,y) \, \mathrm{d}x\mathrm{d}y = \int_a^b \left\{ \int_{L(x)}^{U(x)} f(x,y) \, \mathrm{d}y \right\} \, \mathrm{d}x \,.$$

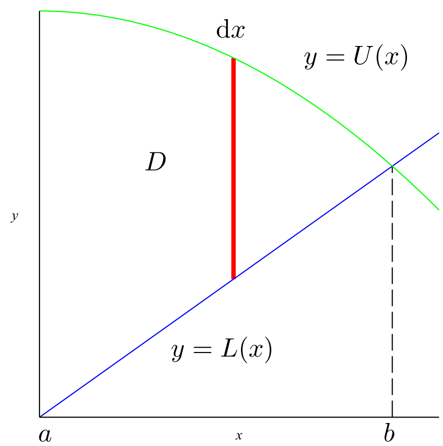

FIGURE 4. A domain of integration D in the xy-plane. The domain is bounded by the vertical lines $y = a$ and $y = b$, and by the two curves $y = L(x)$ and $y = U(x)$. Vertical slicing is appropriate for computing integrals over such a domain.

We work out b by solving the equation

$$U(x) = L(x)\,.$$

Example 4.1. *Figure 4 was actually produced by using $a = 0$ and*

$$L(x) = x,\;\; U(x) = 1 - x^2\,.$$

The value of b is then easily found:

$$b = \frac{\sqrt{5} - 1}{2}\,.$$

So, for instance, for the integrand $f(x, y) = y$, we find

$$\begin{aligned}\iint_D y\,dx\,dy &= \int_0^b \left\{ \int_x^{1-x^2} y\,dy \right\}\,dx = \int_0^b \frac{y^2}{2}\Big|_x^{1-x^2}\,dx \\ &= \int_0^b \frac{1}{2}\left[(1-x^2)^2 - x^2\right]\,dx = \int_0^b \frac{1}{2}\left[1 - 3x^2 + x^4\right]\,dx \\ &= \frac{1}{2}\left[x - x^3 + \frac{x^5}{5}\right]\Big|_0^b = \frac{1}{2}\left[b - b^3 + \frac{b^5}{5}\right]\,.\end{aligned}$$

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, McGraw–Hill, 1999.
2. E. Hairer and G. Wanner, *Analysis by its History*, Springer-Verlag, New-York, 1996.
3. `http://en.wikipedia.org/wiki/File:Volume_under_surface.png`