

MATH11007 NOTES 21: POLAR COORDINATES.

1. WHAT ARE THEY?

Every point in $(x, y) \in \mathbb{R}^2$ may be expressed in polar coordinates (r, θ) as follows

$$r := \sqrt{x^2 + y^2} \text{ and } \theta := \arctan \frac{y}{x}.$$

See Figure 1.

These coordinates occur naturally in many situations. We shall discuss how to

- (1) Plot a curve expressed in polar coordinates.
- (2) Compute the curvature.
- (3) Compute the gradient of a function of the polar coordinates.
- (4) Compute an integral over a domain expressed in polar coordinates.

2. PLOTTING CURVES EXPRESSED IN POLAR COORDINATES

A *curve* is the geometric expression of a set of points whose coordinates are related in some way. We are used to plotting curves arising from a relationship of the form

$$y = f(x).$$

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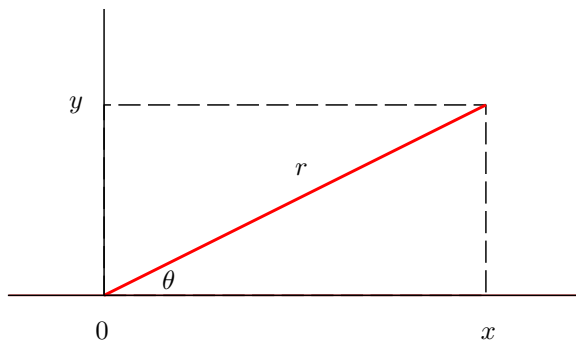


FIGURE 1. Polar coordinates for points in the xy -plane.

But how does one draw, in the xy plane, curves for which the relationship is expressed in the polar form

$$r = R(\theta)?$$

One method is to express r and θ in terms of x and y . Setting

$$g(x, y) = x^2 + y^2 - R^2 \left(\arctan \frac{y}{x} \right),$$

the problem is then reduced to that of plotting a level curve of the function g .

Example 2.1. Draw the curve defined by

$$r = c > 0$$

in the xy -plane.

Solution:

$$g(x, y) = x^2 + y^2 - c^2.$$

The curve is the circle of radius c centered at the origin.

Another, less systematic method is to work out the xy coordinates of a few points on the curve, draw these points, and interpolate between them.

Example 2.2. Draw the curve defined by

$$r = 1 + \cos \theta$$

in the xy -plane.

Solution: We have

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
r	2	1	0	1

The result is shown in Figure 2.

A third (lazy) method consists of using Maple! The plot in Figure 2 was obtained by using the command

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plots[polarplot](1+cos(theta),theta=0..2*Pi,thickness=2) ;
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3. COMPUTING THE CURVATURE

Recall that for a curve expressed in the cartesian form $y = y(x)$, the curvature κ is given by the formula

$$(3.1) \quad \kappa = \frac{|y''|}{\left[1 + (y')^2\right]^{\frac{3}{2}}}.$$

Now, suppose that the curve is given in the polar form

$$r = R(\theta).$$

How does one compute its curvature? In Notes 9, we saw that, if the curve is given in the parametric form

$$x = x(t) \quad \text{and} \quad y = y(t),$$

then

$$\kappa = \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\left[\dot{x}^2 + \dot{y}^2\right]^{\frac{3}{2}}}.$$

Now, using $t = \theta$, the curve can be expressed in the parametric form

$$x(t) = r \cos t = R(t) \cos t \quad \text{and} \quad y(t) = r \sin t = R(t) \sin t.$$

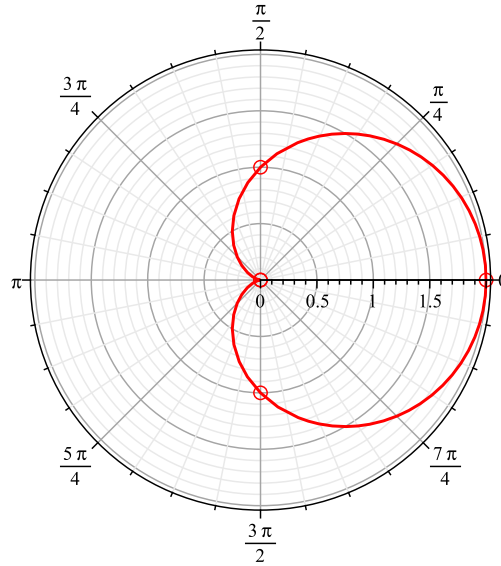


FIGURE 2. Plot of the curve of equation $r = 1 + \cos \theta$ in the xy plane.

Hence (check this!) we deduce

$$(3.2) \quad \kappa = \frac{|R(\theta)^2 + 2[R'(\theta)]^2 - R(\theta)R''(\theta)|}{\{R^2(\theta) + [R'(\theta)]^2\}^{\frac{3}{2}}}.$$

Example 3.1. Consider the curve of equation

$$r = 1 + \cos \theta =: R(\theta).$$

Then

$$R' = -\sin \theta \quad \text{and} \quad R'' = -\cos \theta.$$

We deduce from Equation (3.2)

$$\begin{aligned} \kappa &= \frac{|1 + 2 \cos \theta + \cos^2 \theta + 2 \sin^2 \theta + \cos \theta + \cos^2 \theta|}{\{1 + 2 \cos \theta + 1\}^{\frac{3}{2}}} \\ &= \frac{|3 + 3 \cos \theta|}{\{2 + 2 \cos \theta\}^{\frac{3}{2}}} = \frac{3}{2\sqrt{2}} \frac{1}{\sqrt{1 + \cos \theta}}. \end{aligned}$$

4. COMPUTING THE GRADIENT

Suppose that

$$f(x, y) = u(r, \theta).$$

Then

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} \\ \frac{\partial f}{\partial y} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial y}. \end{aligned}$$

Now, by definition, $r = \sqrt{x^2 + y^2}$ and so

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{r} = \cos \theta,$$

$$\frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2 + y^2}} = \frac{y}{r} = \sin \theta.$$

Also, $\theta = \arctan(y/x)$, and so

$$\frac{\partial \theta}{\partial x} = \frac{-y/x^2}{1 + y^2/x^2} = \frac{-y}{x^2 + y^2} = \frac{-\sin \theta}{r},$$

$$\frac{\partial \theta}{\partial y} = \frac{x/x^2}{1 + y^2/x^2} = \frac{x}{x^2 + y^2} = \frac{\cos \theta}{r}.$$

Putting these results together, we have found

$$\frac{\partial f}{\partial x} = \cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta}$$

and

$$\frac{\partial f}{\partial y} = \sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta}.$$

5. THE CHANGE OF VARIABLES FORMULA

We begin with a motivating

Example 5.1. *Let*

$$R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

and compute

$$\iint_R e^{-x^2 - y^2} dx dy.$$

Solution?: The problem is symmetric in x and y , so we may as well use horizontal slicing. Then

$$\iint_R e^{-x^2 - y^2} dx dy = \int_{-1}^1 \left\{ e^{-y^2} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} e^{-x^2} dx \right\} dy.$$

But we do not know a primitive for the inner integral.

In this example, it would seem more natural to use polar coordinates to express the integral. In other words, we would like to make the substitution

$$\iint_R f(x, y) dx dy \stackrel{(x,y)=(r \cos \theta, r \sin \theta)}{=} \quad ?$$

5.1. Review of the one-variable case. In one-variable calculus, we often resort to making substitutions in order to find primitives. Recall the idea:

$$\int_{g(a)}^{g(b)} f(x) dx \stackrel{x=g(u)}{\downarrow} \int_a^b (f \circ g)(u) g'(u) du.$$

The formula

$$(5.1) \quad dx = g'(u) du$$

expresses the following: the infinitesimal length du is “stretched” by the transformation g by a factor $g'(u)$. Why is the “stretch factor” precisely of this form? Let h be small. By Taylor’s theorem,

$$g(u+h) = g(u) + g'(u)h + \text{H.O.T.}$$

Hence an interval of infinitesimal length du in the u variable is mapped by g to the interval

$$[g(u), g(u+du)] = [g(u), g(u) + g'(u)du]$$

whose length is $g'(u)du$. The integral is obtained by multiplying each of these infinitesimal lengths by the appropriate value of f and summing over them.

5.2. The case of two variables. Suppose we make the substitution

$$(x, y) = \mathbf{g}(u, v) := (g_1(u, v), g_2(u, v)).$$

How does the infinitesimal area $dudv$ change under \mathbf{g} ?

Let h and k be small. The two-variable version of Taylor’s theorem gives

$$g_1(u+h, v+k) = g_1(u, v) + \frac{\partial g_1}{\partial u}(u, v)h + \frac{\partial g_1}{\partial v}(u, v)k + \text{H.O.T.}$$

and

$$g_2(u+h, v+k) = g_2(u, v) + \frac{\partial g_2}{\partial u}(u, v)h + \frac{\partial g_2}{\partial v}(u, v)k + \text{H.O.T.}$$

We can put this in the matrix form

$$\begin{pmatrix} g_1(u+h, v+k) \\ g_2(u+h, v+k) \end{pmatrix} = \begin{pmatrix} g_1(u, v) \\ g_2(u, v) \end{pmatrix} + J_{\mathbf{g}} \begin{pmatrix} h \\ k \end{pmatrix} + \text{H.O.T.}$$

where

$$(5.2) \quad J_{\mathbf{g}} := \begin{pmatrix} \frac{\partial g_1}{\partial u}(u, v) & \frac{\partial g_1}{\partial v}(u, v) \\ \frac{\partial g_2}{\partial u}(u, v) & \frac{\partial g_2}{\partial v}(u, v) \end{pmatrix}$$

is the so-called *Jacobian matrix* of the transformation \mathbf{g} .

This calculation shows the following: in the neighbourhood of (u, v) , \mathbf{g} is essentially a *linear* transformation, and the matrix corresponding to this transformation is the Jacobian matrix of \mathbf{g} . In the one-variable case, the transformation g mapped intervals to intervals. In the two-variable case, however, it is not true that rectangles are mapped to rectangles. Rather, linear transformations map rectangles to parallelograms. Since \mathbf{g} is essentially equivalent to the linear transformation $J_{\mathbf{g}}$ in a neighbourhood of (u, v) , it follows that the infinitesimal rectangle $[u, u+du] \times [v, v+dv]$ is mapped to the parallelogram shown in Figure 3.

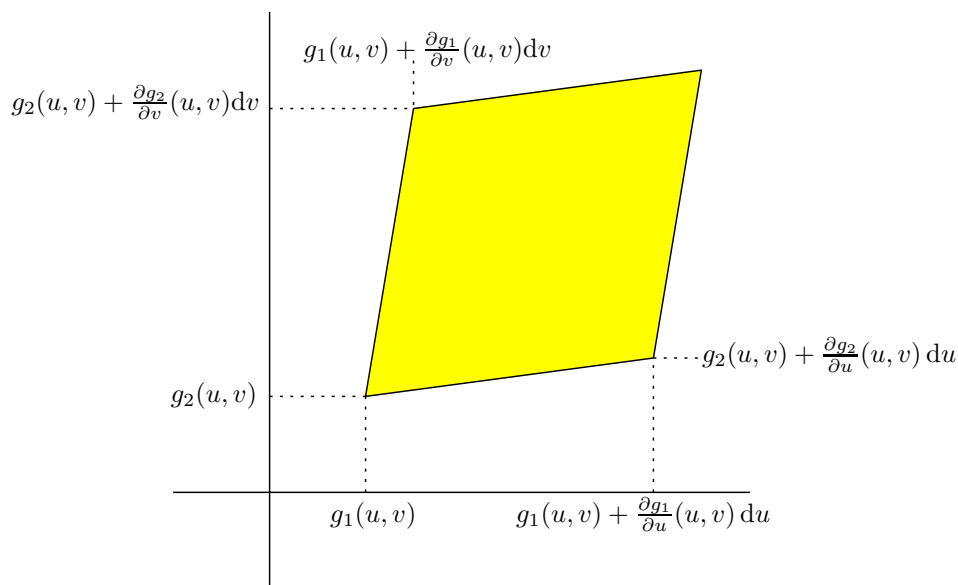


FIGURE 3. Image of the rectangle $[u, u + du] \times [v, v + dv]$ by the map \mathbf{g} .

Now, recall the interpretation of the determinant of a 2×2 matrix in terms of the area of the parallelogram generated by its columns. The area of the parallelogram is related to the area of the rectangle by the formula

$$(5.3) \quad \text{area of parallelogram} = |\det J_{\mathbf{g}}| \, du \, dv.$$

So we have

$$(5.4) \quad \iint_{\mathbf{g}(D)} f(x, y) \, dx \, dy \stackrel{(x, y) = \mathbf{g}(u, v)}{=} \iint_D (f \circ \mathbf{g})(u, v) |\det J_{\mathbf{g}}| \, du \, dv.$$

6. APPLICATIONS OF THE CHANGE OF VARIABLES FORMULA

We apply this formula to the particular substitution

$$(x, y) = (r \cos \theta, r \sin \theta).$$

Here (u is r and v is θ)

$$\mathbf{g}(r, \theta) = (r \cos \theta, r \sin \theta).$$

To apply Formula (5.4), we need to compute the determinant of the Jacobian matrix

$$J_{\mathbf{g}} := \begin{pmatrix} \frac{\partial g_1}{\partial r}(r, \theta) & \frac{\partial g_1}{\partial \theta}(r, \theta) \\ \frac{\partial g_2}{\partial r}(r, \theta) & \frac{\partial g_2}{\partial \theta}(r, \theta) \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}.$$

Hence

$$|J_{\mathbf{g}}| = r.$$

The formula for changing to polar coordinates is therefore

$$(6.1) \quad \iint_{\mathbf{g}(D)} f(x, y) \, dx \, dy = \iint_D f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.$$

Example 6.1. We return to our previous example: the domain of integration is

$$R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

and we want to compute

$$\iint_R e^{-x^2-y^2} dx dy.$$

The substitution

$$(x, y) = \mathbf{g}(r, \theta) := (r \cos \theta, r \sin \theta)$$

enables us to write

$$R = \mathbf{g}(D) \text{ where } D := [0, 1] \times [0, 2\pi].$$

Then, using (6.1), we obtain

$$\begin{aligned} \iint_R e^{-x^2-y^2} dx dy &= \iint_D e^{-r^2} r dr d\theta = \int_0^{2\pi} \left\{ \int_0^1 e^{-r^2} r dr \right\} d\theta \\ &= \int_0^{2\pi} \left\{ -\frac{1}{2} e^{-r^2} \Big|_0^1 \right\} d\theta = \frac{1}{2} \left[1 - \frac{1}{e} \right] \int_0^{2\pi} d\theta = \pi \left[1 - \frac{1}{e} \right]. \end{aligned}$$

A miracle!

Example 6.2. Compute

$$I := \int_{-\infty}^{\infty} e^{-x^2} dx.$$

At first sight, this has nothing to do with double integrals... However

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-x^2} dx \times \int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx \times \int_{-\infty}^{\infty} e^{-y^2} dy \\ &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} e^{-x^2-y^2} dx \right\} dy = \iint_{\mathbb{R}^2} e^{-x^2-y^2} dx dy. \end{aligned}$$

We make the substitution

$$(x, y) = (r \cos \theta, r \sin \theta).$$

Then \mathbb{R}^2 is the image under this substitution of the rectangle $[0, \infty) \times [0, 2\pi]$. Hence

$$\begin{aligned} I^2 &= \iint_{\mathbb{R}^2} e^{-x^2-y^2} dx dy = \iint_{[0, \infty) \times [0, 2\pi]} e^{-r^2} r dr d\theta \\ &= \int_0^{2\pi} \left\{ \int_0^{\infty} e^{-r^2} r dr \right\} d\theta = \int_0^{2\pi} \left\{ -\frac{1}{2} e^{-r^2} \Big|_0^{\infty} \right\} d\theta \\ &= \frac{1}{2} \int_0^{2\pi} d\theta = \pi. \end{aligned}$$

We deduce

$$I = \sqrt{\pi}.$$

Example 6.3. Find the area inside the curve $r = 1 + \cos \theta$ and outside the circle $r = 1$.

Solution: Denote by R the region whose area we seek. Make the substitution

$$(x, y) = (r \cos \theta, r \sin \theta)$$

and look at Figure 1. The region R is the image under this substitution of the domain

$$D = \{(r, \theta) : 1 \leq r \leq 1 + \cos \theta, -\pi/2 \leq \theta \leq \pi/2\}.$$

Hence

$$\begin{aligned} |R| &= \iint_R dx dy = \iint_D r dr d\theta = \int_{-\pi/2}^{\pi/2} \left\{ \int_1^{1+\cos \theta} r dr \right\} d\theta \\ &= \int_{-\pi/2}^{\pi/2} \left\{ \frac{r^2}{2} \Big|_1^{1+\cos \theta} \right\} d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \{2 \cos \theta + \cos^2 \theta\} d\theta \\ &= \int_0^{\pi/2} \{2 \cos \theta + \cos^2 \theta\} d\theta = \left[2 \sin \theta + \frac{\theta}{2} + \frac{\sin(2\theta)}{4} \right] \Big|_0^{\pi/2} = 2 + \pi/4. \end{aligned}$$

REFERENCES

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, Chapter 41, Mc-Graw-Hill, 1999.
2. Serge Lang, *Calculus of Several Variables, Second Edition*, Chapter VII, §3, Addison-Wesley, Reading, 1979.