MATH11007 NOTES 22: TRIPLE INTEGRALS, SPHERICAL COORDINATES.

1. Volumes and hypervolumes

Consider the problem of computing the volume of the "box"

$$D = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]\,.$$

Since we are very good at computing areas, we may compute the volume of the box by slicing "along the z axis"; see Figure 1.

Fix $z \in [0, 1]$ and consider the contribution of a slice at z, of infinitesimal thickness $\mathrm{d}z$; its volume is

$$(\text{surface area of the slice}) \times (\text{thickness of the slice}) = \iint_R \mathrm{d}x\mathrm{d}y \times \mathrm{d}z$$

where

$$R := [a_1, b_1] \times [a_2, b_2]\,.$$

The total volume of the box is obtained by summing over every value of z:

$$\text{volume} = \int_{a_3}^{b_3} \left\{ \iint_R \mathrm{d}x\mathrm{d}y \right\} \mathrm{d}z\,.$$

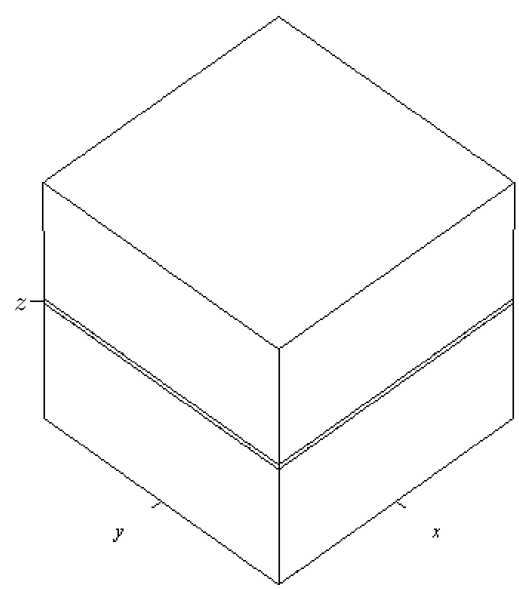

Figure 1. Slicing the unit cube along the z axis.

This is an example of a *triple integral*. We could express the result in the equivalent form

$$\iiint_D f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{a_3}^{b_3} \left\{ \iint_R f(x,y,z)\,\mathrm{d}x\mathrm{d}y \right\} \mathrm{d}z$$

with $f \equiv 1$.

There is no reason to confine ourselves to the case where the integrand f is identically one. For a general integrand, the triple integral on the left may be thought of as a "hypervolume"— something like a volume, but in four-dimensional space. When D is a box and R the rectangle defined earlier, the equality between the triple integral and the simple integral (of a double integral!) on the right gives a method for computing the triple integral by slicing along the z axis.

2. Triple integrals over more general domains

Triple integrals may be defined more generally on other three-dimensional regions. Consider a region defined by

$$D = \left\{(x,y,z) \in \mathbb{R}^3 :\ a \le x \le A(y,z),\, b \le y \le B(z),\, c \le z \le C\right\}$$

where A is a function of y and z, B is a function of z, and C is a constant. Then the triple integral over D may be written in terms of simple integrals as follows:

$$\iiint_D f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_c^C \left\{ \int_b^{B(z)} \left[\int_a^{A(y,z)} f(x,y,z)\,\mathrm{d}x \right] \mathrm{d}y \right\} \mathrm{d}z\,. \tag{2.1}$$

Example 2.1. *Let T be the tetrahedron of vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$; see Figure 2. Then*

$$T = \left\{(x,y,z) \in \mathbb{R}^3 :\ 0 \le x \le 1-y-z,\, 0 \le y \le 1-z,\, 0 \le z \le 1\right\}$$

and so the formula (2.1) gives

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \int_0^1 \left\{ \int_0^{1-z} \left[\int_0^{1-y-z} f(x,y,z)\,dx \right] dy \right\} dz\,.$$

In particular, the volume $|T|$ *of the tetrahedron is*

$$\begin{aligned}|T| &= \int_0^1 \left\{ \int_0^{1-z} \left[\int_0^{1-y-z} dx \right] dy \right\} dz \\ &= \int_0^1 \left\{ \int_0^{1-z} [1-y-z]\ dy \right\} dz = \int_0^1 \left\{ \left[y(1-z) - \frac{y^2}{2} \right] \Big|_0^{1-z} \right\} dz \\ &= \int_0^1 \left\{ \frac{(1-z)^2}{2} \right\} dz = -\frac{(1-z)^3}{6}\Big|_0^1 = \frac{1}{6}\,.\end{aligned}$$

The integrand need not be unity. For instance,

$$\begin{aligned}\iiint_T y\,dx\,dy\,dz &= \int_0^1 \left\{ \int_0^{1-z} \left[\int_0^{1-y-z} y\,dx \right] dy \right\} dz \\ &= \int_0^1 \left\{ \int_0^{1-z} \left[(1-z)y - y^2\right]\ dy \right\} dz = \int_0^1 \left\{ \left[(1-z)\frac{y}{2} - \frac{y^3}{3} \right] \Big|_0^{1-z} \right\} dz \\ &= \int_0^1 \left\{ \frac{(1-z)^3}{6} \right\} dz = -\frac{(1-z)^4}{24}\Big|_0^1 = \frac{1}{24}\,.\end{aligned}$$

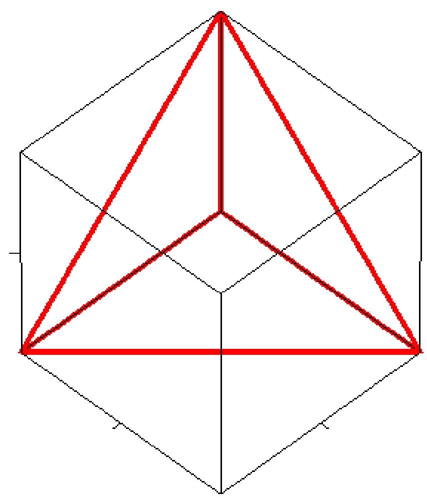

FIGURE 2. A tetrahedral domain.

3. SPHERICAL COORDINATES

Consider the problem of computing the volume of a sphere S of unit radius. This volume equals

$$\iiint_S \mathrm{d}x\mathrm{d}y\mathrm{d}z = 8\int_0^1 \left\{ \int_0^{\sqrt{1-z^2}} \left[\int_0^{\sqrt{1-y^2-z^2}} \mathrm{d}x \right] \mathrm{d}y \right\} \mathrm{d}z\,.$$

We have already mentioned the *spherical coordinates* ρ, θ and φ; see Figure 3. They are connected to the cartesian coordinates via

$$(x,y,z) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,. \tag{3.1}$$

To compute a triple integral over a sphere, it would seem more natural to use spherical coordinates. So we would like to make the substitution

$$\iiint_S f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \overset{(x,y,z)=(\rho\sin\varphi\cos\theta,\rho\sin\varphi\sin\theta,\rho\cos\varphi)}{\overset{\downarrow}{=}} \quad ?$$

For this purpose, we require the following extension of the result we obtained earlier in the two-variable case: suppose we make the substitution

$$(x,y,z) = \mathbf{g}(u,v,w) := (g_1(u,v,w), g_2(u,v,w), g_3(u,v,w))\,.$$

Then

$$\begin{aligned} \iiint_{\mathbf{g}(D)} & f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ & \overset{(x,y,z)=\mathbf{g}(u,v,w)}{\overset{\downarrow}{=}} \iiint_D (f\circ\mathbf{g})(u,v,w)\ |\det J_{\mathbf{g}}|\ \mathrm{d}u\mathrm{d}v\mathrm{d}w \end{aligned} \tag{3.2}$$

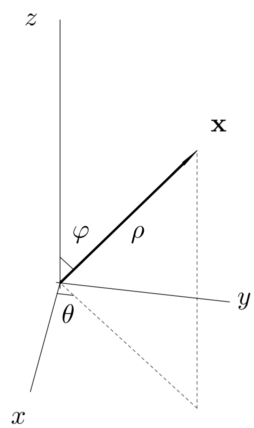

FIGURE 3. The spherical coordinates ρ, θ and φ of the vector $\mathbf{x}$ in three dimensions.

where

$$J_{\mathbf{g}} := \begin{pmatrix} \frac{\partial g_1}{\partial u}(u,v,w) & \frac{\partial g_1}{\partial v}(u,v,w) & \frac{\partial g_1}{\partial w}(u,v,w) \\ \frac{\partial g_2}{\partial u}(u,v,w) & \frac{\partial g_2}{\partial v}(u,v,w) & \frac{\partial g_2}{\partial w}(u,v,w) \\ \frac{\partial g_3}{\partial u}(u,v,w) & \frac{\partial g_3}{\partial v}(u,v,w) & \frac{\partial g_3}{\partial w}(u,v,w) \end{pmatrix} \tag{3.3}$$

is the Jacobian matrix of the transformation $\mathbf{g}$.

We now apply this general formula to the particular substitution (3.1). Here u is ρ, v is φ and $w = \theta$. Furthermore

$$\mathbf{g}(\rho,\varphi,\theta) = (\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\,.$$

To apply Formula (3.2), we need to compute the determinant of the Jacobian matrix

$$J_{\mathbf{g}} := \begin{pmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{pmatrix}\,.$$

Using the third row to develop the determinant, we find

$$\begin{aligned} \det J_{\mathbf{g}} &= \cos\varphi \begin{vmatrix} \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix} + \rho\sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix} \\ &= \cos\varphi[\rho^2\cos\varphi\sin\varphi] \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} + \rho\sin\varphi[\rho\sin^2\varphi] \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \\ &= \cos\varphi[\rho^2\cos\varphi\sin\varphi] + \rho\sin\varphi[\rho\sin^2\varphi] = \rho^2\sin\varphi[\cos^2\varphi + \sin^2\varphi]\,. \end{aligned}$$

Hence

$$\det J_{\mathbf{g}} = \rho^2\sin\varphi\,.$$

The formula for changing to spherical coordinates is therefore

$$(3.4)\quad \iiint_{\mathbf{g}(D)} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_D f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\,\rho^2\sin\varphi\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta\,.$$

Example 3.1. *Let us calculate the volume of S, the unit sphere. The substitution*

$$(x,y,z) = \mathbf{g}(\rho,\varphi,\theta) := (\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)$$

enables us to write

$$S = \mathbf{g}(D) \ \ \textit{where} \ \ D := [0,1]\times[0,\pi]\times[0,2\pi]\,.$$

Then, using (3.4), we obtain

$$\begin{aligned}\iiint_S dx\,dy\,dz &= \iiint_D \rho^2\sin\varphi\, d\rho d\varphi d\theta = \int_0^{2\pi}\left\{\int_0^{\pi}\left[\int_0^1 \rho^2\sin\varphi\, d\rho\right] d\varphi\right\} d\theta \\ &= \int_0^{2\pi}\left\{\int_0^{\pi}\left[\frac{\rho^3}{3}\sin\varphi\Big|_0^1\right] d\varphi\right\} d\theta = \int_0^{2\pi}\left\{\int_0^{\pi}\left[\frac{1}{3}\sin\varphi\right] d\varphi\right\} d\theta \\ &\int_0^{2\pi}\left\{-\left[\frac{1}{3}\cos\varphi\right]\Big|_0^{\pi}\right\} d\theta = \int_0^{2\pi}\left\{\frac{2}{3}\right\} d\theta = \frac{4\pi}{3}\,.\end{aligned}$$

Example 3.2. *Consider the sphere of unit radius centered at $(0,0,1)$, and the cone of equation*

$$z^2 = x^2 + y^2\,.$$

Find the volume above the cone and inside the sphere.

Solution: *The volume may be expressed as a triple integral over a certain region. It seems advisable to use spherical coordinates to describe the region. The equation of the sphere is*

$$1 \ge x^2 + y^2 + (z-1)^2 = x^2 + y^2 + z^2 - 2z + 1\,.$$

In spherical coordinates, this gives

$$\rho \le 2\cos\varphi\,.$$

On the other hand, for the points above the cone,

$$z^2 \ge x^2 + y^2\,.$$

In spherical coordinates, this is

$$\rho^2\cos^2\varphi \ge \rho^2\sin^2\varphi\,.$$

This simplifies to give

$$\tan^2\varphi \le 1$$

or, equivalently, since we are only interested in the range $\varphi \in [0,\pi/2]$,

$$0 \le \varphi \le \pi/4\,.$$

We conclude that the region whose volume is sought is given (in spherical coordinates) by

$$D := \{(\rho,\theta,\varphi)\ :\ 0\le\rho\le 2\cos\varphi\,,\ 0\le\theta\le 2\pi\,,\ 0\le\varphi\le\pi/4\}\,.$$

Hence the volume is

$$\int_0^{\pi/4} \left\{ \int_0^{2\pi} \left[\int_0^{2\cos\varphi} \rho^2 \sin\varphi \, d\rho \right] d\theta \right\} d\varphi = \int_0^{\pi/4} \left\{ \int_0^{2\pi} \left[\frac{\rho^3}{3} \sin\varphi \Big|_0^{2\cos\varphi} \right] d\theta \right\} d\varphi$$
$$= \int_0^{\pi/4} \left\{ \int_0^{2\pi} \left[\frac{8}{3} \cos^3\varphi \, \sin\varphi \right] d\theta \right\} d\varphi = \int_0^{\pi/4} \left\{ \frac{16\pi}{3} \cos^3\varphi \, \sin\varphi \right\} d\varphi$$
$$= -\frac{4\pi}{3} \cos^4\varphi \Big|_0^{\pi/4} = \frac{4\pi}{3} \left(1 - \frac{1}{4} \right) = \pi \, .$$

4. Application: the center of mass of a body

Consider a string extending from a point $x = a$ to a point b, made of some material whose *density* (mass per unit length) we denote by ϱ. If the material is inhomogeneous, ϱ will depend on the position x along the string. The total mass of the string is

$$m := \int_a^b \varrho(x) \, \mathrm{d}x \, .$$

The *center of mass* of the string is then defined by

$$\overline{x} := \frac{1}{m} \int_a^b x \varrho(x) \, \mathrm{d}x \, .$$

These definitions extend readily to higher-dimensional bodies occupying a region D in $\mathbb{R}^d$: the total mass is given by

$$m := \int \cdots \int_D \varrho(\mathbf{x}) \, \mathrm{d}x_1 \cdots \mathrm{d}x_d \tag{4.1}$$

and the jth coordinate of the center of mass is defined by

$$\overline{x}_j := \frac{1}{m} \int \cdots \int_D x_j \, \varrho(\mathbf{x}) \, \mathrm{d}x_1 \cdots \mathrm{d}x_d \, . \tag{4.2}$$

Example 4.1. *Let D be the region represented in polar coordinates by*

$$D = \{(r, \theta) \, : \, 0 \le r \le 1, \; 0 \le \theta \le \pi/2\} \, .$$

This is the quarter-circle of unit radius. The total mass of a body of uniform density ($\varrho \equiv 1$) is

$$m = \int_0^{\pi/2} \left\{ \int_0^1 r \, dr \right\} d\theta = \int_0^{\pi/2} \left\{ \frac{1}{2} r^2 \Big|_0^1 \right\} d\theta = \int_0^{\pi/2} \frac{1}{2} \, d\theta = \pi/4 \, .$$

Let us calculate the center of gravity: we have

$$\overline{x} = \int_0^{\pi/2} \left\{ \int_0^1 r^2 \cos\theta \, dr \right\} d\theta = \int_0^{\pi/2} \left\{ \frac{1}{3} r^3 \cos\theta \Big|_0^1 \right\} d\theta$$
$$= \int_0^{\pi/2} \frac{1}{3} \cos\theta \, d\theta = \frac{1}{3} \sin\theta \Big|_0^{\pi/2} = 1/3 \, .$$

Also

$$\overline{y} = \int_0^{\pi/2} \left\{ \int_0^1 r^2 \sin\theta \, dr \right\} d\theta = \int_0^{\pi/2} \left\{ \frac{1}{3} r^3 \sin\theta \Big|_0^1 \right\} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3} \sin\theta \, d\theta = -\frac{1}{3} \cos\theta \Big|_0^{\pi/2} = 1/3\,.$$

So the center of gravity of the body is

$$(\overline{x},\, \overline{y}) = \frac{4}{3\pi}(1,\, 1)\,.$$

Example 4.2. *Let C be the cone of height h whose base is the unit disk $x^2+y^2 \leq 1$. Assume that the density is uniform, i.e. $\varrho \equiv 1$. Find the center of mass.*

Solution*: The equation of the cone (in cartesian coordinates) is*

$$\left(\frac{h-z}{h} \right)^2 = x^2 + y^2\,.$$

If the cone were upside down, there would be some advantage in using spherical coordinates …but this is not the case and so we stick to cartesian coordinates for the moment. The total mass is given by

$$m = \int_0^{\frac{h-z}{h}} \left\{ \iint_{R(z)} dx\, dy \right\} dz$$

where

$$R(z) := \left\{ (x,y) \in \mathbb{R}^2 \;:\; x^2 + y^2 \leq \left(\frac{h-z}{h} \right)^2 \right\}\,.$$

At this point, we switch to polar *coordinates to evaluate the double integral over $R(z)$. This yields*

$$\iint_{R(z)} dx\, dy = \int_0^{2\pi} \left[\int_0^{\frac{h-z}{h}} r\, dr \right] d\theta = \int_0^{2\pi} \left[\frac{r^2}{2} \Big|_0^{\frac{h-z}{h}} \right] = 2\pi \frac{(h-z)^2}{2h^2}\,.$$

We deduce

$$m = \int_0^h \pi \left(\frac{h-z}{h} \right)^2 dz = \pi h \int_0^1 u^2\, du = \frac{\pi h}{3}\,.$$

Then, using the same approach,

$$m\overline{x} = \int_0^h \left\{ \iint_{R(z)} x\; dx\, dy \right\} dz = \int_0^h \left\{ \int_0^{2\pi} \left[\int_0^{\frac{h-z}{h}} r^2 \cos\theta\; dr \right] d\theta \right\} dz$$

$$= \int_0^h \left\{ \int_0^{2\pi} \left[\frac{1}{3} \left(\frac{h-z}{h} \right)^3 \cos\theta \right] d\theta \right\} dz = \int_0^h \left\{ \left[\frac{1}{3} \left(\frac{h-z}{h} \right)^3 \sin\theta \right] \Big|_0^{2\pi} \right\} dz$$

$$= \int_0^h \{0\}\; dz = 0\,.$$

By symmetry

$$m\overline{y} = 0\,.$$

Finally

$$m\overline{z} = \int_0^h z\pi\left(\frac{h-z}{h}\right)^2\, dz = \frac{\pi}{h^2}\int_0^h z\,(h-z)^2\, dz$$
$$= \frac{\pi}{h^2}\int_0^h z^3 - 2hz^2 + h^2 z\, dz = \pi h^2\left[\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right] = \pi\frac{h^2}{12}\,.$$

Thus, we have found

$$(\overline{x},\, \overline{y},\, \overline{z}) = \frac{h}{4}\,(0,\, 0,\, 1)\ .$$

Example 4.3. *Find the mass of a cylinder of radius a and height h assuming its density is proportional to the square of the distance from the axis.*

Solution*: we let the axis be the z-axis and assume that the cylinder rests on the xy-plane at the origin. Then*

$$\varrho(x,y,z) = k\left(x^2+y^2\right)$$

for some positive constant k, and so

$$m = \int_0^h \left\{\iint_D k\left(x^2+y^2\right)\, dx\,dy\right\}\, dz$$

where D is the disk of radius a centered at the origin. Using polar coordiantes for the double integral, we obtain

$$m = \int_0^h\left\{\int_0^{2\pi}\left[\int_0^a kr^3\, dr\right] d\theta\right\}\, dz = \int_0^h\left\{\int_0^{2\pi}\left[k\frac{a^4}{4}\right] d\theta\right\}\, dz = kh\pi\frac{a^4}{2}\,.$$

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, Chapter 41, Mc-Graw–Hill, 1999.
2. Serge Lang, *Calculus of Several Variables, Second Edition*, Chapter VII, §3, Addison–Wesley, Reading, 1979.