

## MATH11007 NOTES 5: THE FUNDAMENTAL THEOREM OF CALCULUS

### 1. THE CONCEPT OF PRIMITIVE

**Definition 1.1.** We say that a function  $F : A \rightarrow \mathbb{R}$  is a *primitive* (or an anti-derivative, or an indefinite integral) of the function  $f : A \rightarrow \mathbb{R}$  if  $F$  is differentiable and

$$F' = f.$$

**Example 1.1.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = x^n, \quad n \in \mathbb{N}.$$

Then the following functions are all primitives of  $f$ :

(1)

$$F(x) = \frac{x^{n+1}}{n+1}.$$

(2)

$$F(x) = \frac{x^{n+1}}{n+1} + 10^6.$$

**Example 1.2.** Let  $f : (0, \infty) \rightarrow (0, \infty)$  be defined by

$$f(x) = \sqrt{x}.$$

Then

$$F(x) = \frac{2}{3}x^{3/2} + 1$$

is a primitive.

**Example 1.3.** Let  $f(x) = \sin x$ . Then  $F(x) = -\cos x + \pi$  is a primitive.

**Example 1.4.**  $F(x) = e^x + 1/2$  is a primitive of  $f(x) = e^x$ .

**Example 1.5.**  $F(x) = \arctan x$  is a primitive of  $f(x) = 1/(1+x^2)$ .

**Lemma 1.1.** If  $F_1$  and  $F_2$  are two primitives of the same function  $f : A \rightarrow \mathbb{R}$ , then there exists a constant  $c$  such that, for every  $x \in A$ ,

$$F_1(x) - F_2(x) = c.$$

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*Proof.* Let

$$G(x) = F_1(x) - F_2(x).$$

Then, for every  $x \in A$ ,

$$G'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0.$$

So, for every  $x \in A$ , the slope of the line tangent to the curve of equation

$$y = G(x)$$

is zero. It follows that the curve is a horizontal line, i.e.  $y = c$  for some constant  $c$ .  $\square$

Given  $f$  and a function  $F$ , it is relatively easy to determine whether or not  $F$  is a primitive of  $f$ : it suffices to compute  $F'$ . But if we are given only  $f$ , it is not straightforward to *find* a primitive.

## 2. THE AREA UNDER A CURVE

Let  $f : [a, b] \rightarrow \mathbb{R}$  be *continuous* and consider the curve of equation  $y = f(x)$ ; see Figure 1.

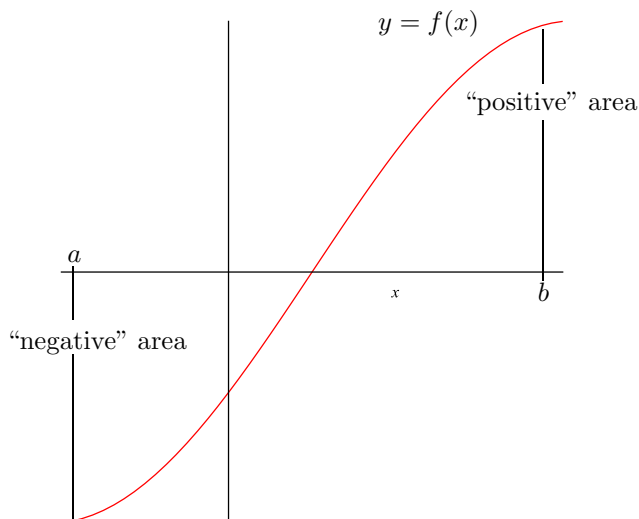


FIGURE 1. The area under a curve.

*Notation .*

$$\int_a^b f(x) dx := \text{area under the curve between } x = a \text{ and } x = b.$$

Some remarks:

- (1) It does not matter what letter one uses to denote the independent variable.

Thus

$$\int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(u) du \quad \text{etc.}$$

(2) Geometrically, it is clear that, for every  $a < c < b$ ,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

(3) We shall use the convention

$$\int_b^a f(x) dx := - \int_a^b f(x) dx.$$

Now, let  $a$  be fixed, but allow the right endpoint to vary between  $a$  and  $b$ : we obtain a function  $I : [a, b] \rightarrow [0, \infty)$  defined by

$$(2.1) \quad I(x) := \int_a^x f(t) dt.$$

This notation may be disconcerting at first sight, but remember that we agreed that we could use any letter for the independent variable, e.g.

$$\int_a^b f(x) dx := \int_a^b f(u) du.$$

Then we are free to use  $x$  for the right endpoint of the interval! This explains the notation in Equation (2.1).

**Theorem 2.1.** *The function  $I$  is a primitive of  $f$ .*

*Proof.* Let  $x_0 \in [a, b]$  and let  $h \neq 0$  be such that  $x_0 + h \in [a, b]$ . We have

$$\begin{aligned} \frac{I(x_0 + h) - I(x_0)}{h} &= \frac{1}{h} \int_a^{x_0+h} f(t) dt - \frac{1}{h} \int_a^{x_0} f(t) dt \\ &= \frac{1}{h} \int_a^{x_0} f(t) dt + \frac{1}{h} \int_{x_0}^{x_0+h} f(t) dt - \frac{1}{h} \int_a^{x_0} f(t) dt = \frac{1}{h} \int_{x_0}^{x_0+h} f(t) dt. \end{aligned}$$

For simplicity, suppose that  $h > 0$  and define

$$m_h(x_0) := \min_{x_0 \leq t \leq x_0+h} f(t) \quad \text{and} \quad M_h(x_0) := \max_{x_0 \leq t \leq x_0+h} f(t).$$

It is then geometrically clear (see Figure 2) that

$$m_h(x_0)h \leq \int_{x_0}^{x_0+h} f(t) dt \leq M_h(x_0)h.$$

After dividing by  $h$ , this gives

$$(2.2) \quad m_h(x_0) \leq \frac{1}{h} \int_{x_0}^{x_0+h} f(t) dt \leq M_h(x_0).$$

Now, since, by hypothesis,  $f$  is continuous at  $x_0$ , we have

$$\lim_{h \rightarrow 0} m_h(x_0) = \lim_{h \rightarrow 0} M_h(x_0) = f(x_0).$$

So we deduce from (2.2) that

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_{x_0}^{x_0+h} f(t) dt = f(x_0).$$

In other words,

$$I'(x_0) = f(x_0).$$

□

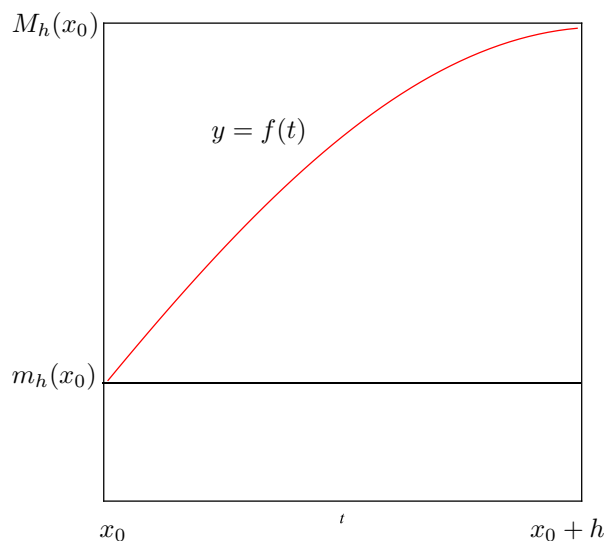


FIGURE 2. Proof of Theorem 2.1: the area under the curve is squeezed between  $m_h(x_0)h$  and  $M_h(x_0)h$ .

**Corollary 2.2.** *Let  $F$  be any primitive of  $f$ . Then*

$$\int_a^b f(x) dx = F(b) - F(a).$$

*Proof.* Let  $F$  be a primitive of  $f$ . Since  $I$  is also a primitive of  $f$ , there exists a constant  $c$  such that, for every  $x \in [a, b]$ ,

$$I(x) = F(x) + c.$$

Then

$$\int_a^b f(x) dx = I(b) - I(a) = [F(b) + c] - [F(a) + c] = F(b) - F(a).$$

□

*Notation .*

$$F(x) = \int f(x) dx$$

means “ $F$  is a primitive of  $f$ ”. Also

$$F(x) \Big|_a^b := F(b) - F(a).$$

**Example 2.1.** *Problem: Compute the area under the parabola of equation  $y = x^2$  between  $x = 0$  and  $x = 1$ .*

*Solution: Set  $f(x) = x^2$ . The function  $F(x) = x^3/3$  is a primitive of  $f$ . So, by the Fundamental Theorem of Calculus, the area is*

$$\int_0^1 f(x) dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}.$$

**Example 2.2.** Problem: *Compute the area of a circle of unit radius.*

Solution: *Set  $f(x) = \sqrt{1-x^2}$ . Then the area is*

$$4 \int_0^1 f(x) dx.$$

*So we look for a primitive of  $f$ !*

### 3. SUBSTITUTIONS

In this section, we consider a very useful technique for finding primitives.

**Theorem 3.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous, and let  $u$  be a differentiable bijection with codomain  $[a, b]$ . Let  $\alpha$  and  $\beta$  be such that*

$$u(\alpha) = a \quad \text{and} \quad u(\beta) = b.$$

*Then*

$$\int_a^b f(u) du = \int_\alpha^\beta (f \circ u)(x)u'(x) dx.$$

*Proof.* Let  $F$  be a primitive of  $f$ . Then, by the chain rule,

$$\frac{d}{dx}F(u(x)) = F'(u(x))u'(x) = f(u(x))u'(x).$$

In other words,  $F(u(x))$  is a primitive of  $(f \circ u)(x)u'(x)$ . Hence, by the Fundamental Theorem of Calculus,

$$\int_a^b f(u) du = F(b) - F(a) = F(u(\beta)) - F(u(\alpha)) = \int_\alpha^\beta (f \circ u)(x)u'(x) dx.$$

□

To illustrate the use of substitutions, let us go back to Example 2.2. We have

$$\begin{aligned} \int_0^1 \sqrt{1-x^2} dx &\stackrel{u=x}{=} \int_0^1 \sqrt{1-u^2} du \\ &\stackrel{u=\cos x}{=} \int_{\frac{\pi}{2}}^0 \sqrt{1-\cos^2 x} (-\sin x) dx = \int_0^{\frac{\pi}{2}} \sin^2 x dx. \end{aligned}$$

Now, it is easy to find a primitive of  $\sin^2 x$ , and we obtain

$$\int_0^1 \sqrt{1-x^2} dx = \left( \frac{x}{2} - \frac{\sin(2x)}{4} \right) \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

So the area of a circle of unit radius is  $\pi$ .

### REFERENCES

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, McGraw-Hill, 1999.